DISPERSIVE ESTIMATES FOR SCHRÖDINGER OPERATORS IN DIMENSION TWO WITH OBSTRUCTIONS AT ZERO ENERGY

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ABSTRACT. We investigate $L^1(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)$ dispersive estimates for the Schrödinger operator $H = -\Delta + V$ when there are obstructions, resonances or an eigenvalue, at zero energy. In particular, we show that the existence of an s-wave resonance at zero energy does not destroy the t^{-1} decay rate. We also show that if there is a p-wave resonance or an eigenvalue at zero energy then there is a time dependent operator F_t satisfying $\|F_t\|_{L^1 \to L^\infty} \lesssim 1$ such that

$$\|e^{itH} P_{ac} - F_t\|_{L^1 \to L^\infty} \lesssim |t|^{-1}, \quad \text{for } |t| > 1.$$

We also establish a weighted dispersive estimate with t^{-1} decay rate in the case when there is an eigenvalue at zero energy but no resonances.

1. INTRODUCTION

Consider the Schrödinger operator $H = -\Delta + V$ in $\mathbb{R}^2$, where V is a real-valued potential. Let P_{ac} be the orthogonal projection onto the absolutely continuous subspace of $L^2(\mathbb{R}^2)$, which is determined by H. In [23], Schlag proved that

$$\|e^{itH} P_{ac}\|_{L^1(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)} \lesssim |t|^{-1}$$

under the decay assumption $|V| \lesssim \langle x \rangle^{-3-}$ and the assumption that zero is neither an eigenvalue nor a resonance of H.

Recall that (see, e.g., [14] or Section 5 below) there is a resonance at zero energy if there is a distributional solution to the equation $H\psi = 0$ where $\psi \notin L^2(\mathbb{R}^2)$ but $\psi \in L^p(\mathbb{R}^2)$ for some $p \in (2, \infty]$. There are two possible cases, either $\psi \in L^\infty(\mathbb{R}^2)$ and $\psi \notin L^p(\mathbb{R}^2)$ for any $p < \infty$ or $\psi \in L^p(\mathbb{R}^2)$ for all $p \in (2, \infty]$. In the case of $\psi \in L^\infty(\mathbb{R}^2)$ only, the resonance is called an s-wave resonance. In the second case, we say there is a p-wave resonance. We say that there is an eigenvalue at zero if $\psi \in L^2(\mathbb{R}^2)$. This definition for resonances differs from the case of dimension $n = 3$ in which ψ lies in weighted L^2 spaces.

We note that in the case of $V \equiv 0$ the function $\psi \equiv 1$ solves $H\psi = 0$ which corresponds to an s-wave resonance. It is important to note that in spite of this obstruction, the free evolution decays in time at the rate t^{-1}.

Much is known about dispersive estimates for the Schrödinger equation when zero is regular. The history goes back to Rauch, [19], who studied the local decay in dimension three. In [19], he noted that in the generic case, i.e. when there may be eigenvalues or resonances, the evolution decays at a rate of $|t|^{-1/2}$ as $t \to \infty$ on exponentially weighted L^2 spaces. In the case when there are no eigenvalues or resonances, it was shown that the decay rate is $|t|^{-3/2}$. Jensen and Kato, [13], improved this result to polynomially weighted L^2 spaces in dimension three, and higher dimensions, [11, 12]. In [13], it was noted that the presence of a zero energy eigenvalue or resonance destroys the $|t|^{-3/2}$ decay even if one projects away from the eigenspace in dimension three.

Local decay estimates in the two dimensional case when zero is regular were studied by Murata in [18]. Murata was able to prove an estimate on weighted L^2 spaces that decays like $t^{-1}(\log t)^{-2}$, which is integrable at infinity. Such estimates have been used in analysis of the stability of certain two-dimensional non-linear equations.

The first result to discuss global decay, $L^1 \to L^\infty$ estimates, was due to Journé, Soffer and Sogge in [16]. Their result relied on the integrability of $t^{-n/2}$ at infinity and is thus restricted to $n \geq 3$. Much is now known in this direction, mainly in dimension three. Rodnianski and Schlag established such estimates in dimension three, [21], in addition to establishing Strichartz estimates. Following from their methods, a great number of results in dimension three followed, particularly [8, 9, 10]. The one dimensional problem was studied by Weder, [25] and Goldberg and Schlag [10]. Also see [5, 27, 7] for global estimates in the three-dimensional case when there is an eigenvalue and/or resonance at zero energy, and [6] for a similar result for the matrix Schrödinger equation.

There have also been studies of the wave-operators in dimension two. In particular Yajima, [26] established that the wave operators are bounded on $L^p(\mathbb{R}^2)$ for $1 < p < \infty$ if zero is regular. The hypotheses on the potential V were relaxed slightly in [15]. This result would imply global dispersive estimates if extended to the full range of p, $1 \leq p \leq \infty$. High frequency dispersive estimates, similar to those obtained in [23] stated as Theorem 1.3 below were obtained by Moulin, [17], under an integrability condition on the potential.

In this paper we investigate $L^1 \to L^\infty$ dispersive estimates in $\mathbb{R}^2$ when zero energy is not a regular point of the spectrum of the operator $H = -\Delta + V$. Our goal is to prove the following theorem.

Theorem 1.1. *Assume that $|V(x)| \lesssim \langle x \rangle^{-\beta}$. If there is only an s-wave resonance at zero energy, then for $\beta > 4$, we have*

$$\|e^{itH} P_{ac}(H)\|_{L^1 \to L^\infty} \lesssim |t|^{-1}.$$

If there is a p-wave resonance or eigenvalue at zero, then for $\beta > 6$, there is a time-dependent operator F_t such that

$$\|e^{itH} P_{ac}(H) - F_t\|_{L^1 \to L^\infty} \lesssim |t|^{-1}, \qquad |t| > 1,$$

with

$$\sup_t \|F_t\|_{L^1 \to L^\infty} \lesssim 1.$$

Note that it is natural to have the t^{-1} decay rate in the case of an s-wave resonance since the free Schrödinger has an s-wave resonance at zero energy. The reason that we can not get any decay in the case of a p-wave resonance or the zero eigenvalue is the behavior of the resolvent around zero energy. In the three dimensional case the resolvent $(H - z^2)^{-1}$ has an expansion of the form

$$(H - z^2)^{-1} = -G_{-2}z^{-2} + G_{-1}z^{-1} + O(1), \ \ z \to 0, \ \Im(z) > 0.$$

The most singular term $G_{-2}z^{-2}$ gives the Riesz projection to zero energy eigenspace. If one projects away from the zero eigenspace, the worst singularity is $\frac{1}{z}$, which allows for $|t|^{-1/2}$ decay as $t \to \pm\infty$, see [5]. However, in the two dimensional case the resolvent expansion around zero contains logarithmic terms. In particular, in the general case of zero energy resonances (even if one projects away the zero energy eigenspace), the most singular term is of the form $\frac{1}{z^2 \log(z)}$, which does not allow for any polynomial decay in t. It may be possible to get a decay of the form $\frac{1}{\log(t)}$ as in [18] but we won't pursue this issue here. However, it is possible to improve this theorem in the case when zero is an eigenvalue but there are no resonances at zero. In particular, we show the following.

Theorem 1.2. *Assume that $|V(x)| \lesssim \langle x \rangle^{-\beta}$ for some $\beta > 11$. If zero is an eigenvalue of $H = -\Delta + V$ and there are neither s-wave nor p-wave resonances at zero, then*

$$\|e^{itH}P_{ac}\|_{L^{1,1+} \to L^{\infty,-1-}} \lesssim |t|^{-1}.$$

Here $L^{1,1+}$ is the weighted L^1 space defined by $L^{1,1+}(\mathbb{R}^2) := \{f : \int_{\mathbb{R}^2} |f(x)|\langle x \rangle^{1+} \, dx < \infty\}$. Similarly, $L^{\infty,-1-} = \{f : \langle x \rangle^{-1-} f \in L^\infty\}$.

Let χ is an even smooth function supported in $[-\lambda_1, \lambda_1]$ and $\chi(x) = 1$ for $|x| < \lambda_1/2$. Let K_{λ_1} be the kernel of $e^{itH}\chi(H)P_{ac}$:

$$(1) \qquad K_{\lambda_1}(x,y) = \frac{1}{\pi i} \int_0^\infty e^{it\lambda^2} \lambda\chi(\lambda)[R_V^+(\lambda^2) - R_V^-(\lambda^2)](x,y)d\lambda,$$

where

$$R_V^\pm(\lambda^2) = R_V(\lambda^2 \pm i0) = (H - (\lambda^2 \pm i0))^{-1}$$

is the perturbed resolvent. By the limiting absorption principle, these boundary values are bounded operators on weighted L^2-spaces, see e.g. [2].

The high energies were studied in [23]:

Theorem 1.3. [23] *Assume that $|V| \lesssim \langle x \rangle^{-2-}$, then for any $\lambda_1 > 0$*

$$\left\|e^{itH}P_{ac} - e^{itH}\chi(H)P_{ac}\right\|_{1 \to \infty} \leq C_{\lambda_1}|t|^{-1}.$$

Therefore, in the proof of Theorem 1.1 and Theorem 1.2, it suffices to obtain the stated bounds for the operator K_{λ_1} for some $\lambda_1 > 0$. Our analysis relies on expansions of the resolvent operator at zero energy following those of [14], also see the previous work in [3, 4]. We repeat part of the argument to obtain more flexible and favorable error bounds for our purposes.

We also note that standard spectral theoretic results for H apply. Under our assumptions we have that the spectrum of H can be expressed as the absolutely continuous spectrum, the interval $[0, \infty)$, and finitely many eigenvalues of finite multiplicity on $(-\infty, 0]$. See [20] for spectral theory and [24] for Birman-Schwinger type bounds.

Our paper is organized as follows. We set out the necessary expansions for the resolvent in Section 2. We then study K_{λ_1} to establish Theorem 1.1 in the case when there is an s-wave resonance at zero in Section 3. In Section 4 we establish Theorem 1.1 in the case of a p-wave resonance or eigenvalue at zero energy. In Section 5 we discuss the spectral structure of $-\Delta + V$ at zero energy. Finally we prove Theorem 1.2 in Section 6.

2. Resolvent expansions around zero energy in the case of an s-wave resonance

In this section, following [14], we obtain resolvent expansions around the threshold $\lambda = 0$ in the case when there is only s-wave resonance at zero (resonance of the first kind, see Definition 2.3 below and the remarks following it). We now introduce some definitions and notation.

Definition 2.1. *We say an operator* $T : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ *with kernel* $T(\cdot, \cdot)$ *is absolutely bounded if the operator with kernel* $|T(\cdot, \cdot)|$ *is bounded from* $L^2(\mathbb{R}^2)$ *to* $L^2(\mathbb{R}^2)$.

It is worth noting that a Hilbert-Schmidt operator is an absolutely bounded operator.

We say that an absolutely bounded operator $T(\lambda)(\cdot, \cdot)$ is $\mathcal{O}_1(\lambda^s)$ if the integral kernel satisfies the following estimates:

$$(2) \qquad \big\| \sup_{0 < \lambda < \lambda_1} \lambda^{-s} |T(\lambda)(\cdot, \cdot)| \big\|_{L^2 \to L^2} \lesssim 1, \qquad \big\| \sup_{0 < \lambda < \lambda_1} \lambda^{1-s} |\partial_\lambda T(\lambda)(\cdot, \cdot)| \big\|_{L^2 \to L^2} \lesssim 1.$$

If only the first bound in (2) holds, we say that $T(\lambda)(\cdot, \cdot)$ is $O(\lambda^s)$. We also note that we can replace λ^{-s} with $f(\lambda)^{-1}$ in (2), in which case we say $T(\lambda)(\cdot, \cdot)$ is $O(f(\lambda))$.

Recall that

$$R_0^\pm(\lambda^2)(x, y) = \pm \frac{i}{4} H_0^\pm(\lambda |x - y|)$$

where $H_0^\pm$ are the Hankel functions of order zero:

$$(3) \qquad\qquad\qquad H_0^\pm(z) = J_0(z) \pm i Y_0(z).$$

From the series expansions for the Bessel functions, see [1], as $z \to 0$ we have

$$(4) \qquad J_0(z) = 1 - \frac{1}{4}z^2 + \frac{1}{64}z^4 + O(z^6) = 1 + O(z^2),$$

$$(5) \qquad Y_0(z) = \frac{2}{\pi}(\log(z/2) + \gamma)J_0(z) + \frac{2}{\pi}\left(\frac{1}{4}z^2 - \frac{3}{128}z^4 + O(z^6)\right)$$

$$(6) \qquad = \frac{2}{\pi}\log(z) + O(1).$$

We also have the following estimates for the derivatives as $z \to 0$

$$(7) \qquad J_0'(z) = O(z), \quad J_0''(z) = O(1), \quad Y_0'(z) = \frac{2}{\pi z} + O(1).$$

Further, for $|z| > 1$, we have the representation (see, *e.g.*, [1])

$$(8) \qquad H_0^\pm(z) = e^{\pm iz}\omega_\pm(z), \quad |\omega_\pm^{(\ell)}(z)| \lesssim (1 + |z|)^{-\frac{1}{2}-\ell}, \quad \ell = 0, 1, 2, \ldots.$$

This implies that for $|z| > 1$

$$(9) \qquad \mathcal{C}(z) = e^{iz}\omega_+(z) + e^{-iz}\omega_-(z), \qquad |\omega_\pm^{(\ell)}(z)| \lesssim (1 + |z|)^{-\frac{1}{2}-\ell}, \quad \ell = 0, 1, 2, \ldots,$$

for any $\mathcal{C} \in \{J_0, Y_0\}$ respectively with different $\omega_\pm$.

Let $U(x) = 1$ if $V(x) \geq 0$ and $U(x) = -1$ if $V(x) < 0$, and let $v = |V|^{1/2}$. We have $V = Uv^2$. We use the symmetric resolvent identity, valid for $\Im\lambda > 0$:

$$(10) \qquad R_V^\pm(\lambda^2) = R_0^\pm(\lambda^2) - R_0^\pm(\lambda^2)vM^\pm(\lambda)^{-1}vR_0^\pm(\lambda^2),$$

where $M^\pm(\lambda) = U + vR_0^\pm(\lambda^2)v$. The key issue in the resolvent expansions is the invertibility of the operator $M^\pm(\lambda)$ for small λ under various spectral assumptions at zero. Below, we obtain expansions of the operator $M^\pm(\lambda)$ around $\lambda = 0$ using the properties of the free resolvent listed above. A similar lemma was proved in [23], however we need to expand the operator further and obtain slightly more general error bounds. The following operators arise naturally in the expansion of $M^\pm(\lambda)$ (see (4), (5))

$$(11) \qquad G_0 f(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x - y|f(y)\,dy,$$

$$(12) \qquad G_1 f(x) = \int_{\mathbb{R}^2} |x - y|^2 f(y)\,dy,$$

$$(13) \qquad G_2 f(x) = \frac{1}{8\pi}\int_{\mathbb{R}^2} |x - y|^2 \log|x - y|f(x)\,dy.$$

Lemma 2.2. *For $\lambda > 0$ define $M^\pm(\lambda) := U + vR_0^\pm(\lambda^2)v$. Let $P = v\langle\cdot, v\rangle\|V\|_1^{-1}$ denote the orthogonal projection onto v. Then*

$$M^\pm(\lambda) = g^\pm(\lambda)P + T + M_0^\pm(\lambda).$$

Here $g^{\pm}(\lambda) = a \ln \lambda + z$ where $a \in \mathbb{R} \backslash \{0\}$ and $z \in \mathbb{C} \backslash \mathbb{R}$, and $T = U + vG_0v$ where G_0 is an integral operator defined in (11). Further, for any $\frac{1}{2} \leq k < 2$,

$$M_0^{\pm}(\lambda) = \mathcal{O}_1(\lambda^k)$$

if $v(x) \lesssim \langle x \rangle^{-\beta}$ for some $\beta > 1 + k$. Moreover,

$$(14) \qquad M_0^{\pm}(\lambda) = g_1^{\pm}(\lambda)vG_1v + \lambda^2 vG_2v + M_1^{\pm}(\lambda).$$

Here G_1, G_2 are integral operators defined in (12), (13), and $g_1^{\pm}(\lambda) = \lambda^2(\alpha \log \lambda + \beta_{\pm})$ where $\alpha \in \mathbb{R} \backslash \{0\}$ and $\beta_{\pm} \in \mathbb{C} \backslash \mathbb{R}$. Further, for any $2 < \ell < 4$,

$$M_1^{\pm}(\lambda) = \mathcal{O}_1(\lambda^{\ell})$$

if $\beta > 1 + \ell$.

Proof. The first part with $k = \frac{1}{2}$ was proven in [23, Lemma 5]. To obtain the expansions recall that, for $\lambda > 0$,

$$R_0^{\pm}(\lambda^2)(x, y) = \pm \frac{i}{4} H_0^{\pm}(\lambda|x - y|).$$

Using the definition of $H_0^{\pm}(z)$, and the expansions (4) and (5) around $z = 0$, we have

$$(15) \quad \pm \frac{i}{4} H_0^{\pm}(z) = \pm \frac{i}{4} J_0(z) - \frac{1}{4} Y_0(z) = -\frac{1}{2\pi} \log(z/2) \pm \frac{i}{4} - \frac{\gamma}{2\pi} + \alpha z^2 \log z + \beta_{\pm} z^2 + O(z^4 \log z)$$

$$(16) \qquad = -\frac{1}{2\pi} \log(z/2) \pm \frac{i}{4} - \frac{\gamma}{2\pi} + O(z^2 \log z)$$

with $\alpha = 1/8\pi$ and $\beta_{\pm} \in \mathbb{C}$. The expansions are now obtained by setting $z = \lambda|x - y|$. In particular, we see

$$(17) \qquad g^{\pm}(\lambda) = -\|V\|_1 \left(\frac{1}{2\pi} \log(\lambda/2) + \frac{1}{2\pi}\gamma \mp \frac{i}{4} \right).$$

Noting that

$$M_0^{\pm}(\lambda) = [U + vR_0^{\pm}(\lambda^2)v] - [g^{\pm}(\lambda)P + U + vG_0v],$$

$$M_1^{\pm}(\lambda) = [U + vR_0^{\pm}(\lambda^2)v] - [g^{\pm}(\lambda)P + U + vG_0v + g_1^{\pm}(\lambda)vG_1v + \lambda^2 vG_2v].$$

Using (16) and (15) for M_0 and M_1 respectively, we obtain for $z = \lambda|x - y| < 1$ that

$$|M_0^{\pm}(\lambda)(x, y)\chi_{\{\lambda|x-y|<1\}}| \lesssim v(x)v(y)(\lambda|x - y|)^2 \log(\lambda|x - y|) \lesssim v(x)v(y)(\lambda|x - y|)^{2-},$$

$$|M_1^{\pm}(\lambda)(x, y)\chi_{\{\lambda|x-y|<1\}}| \lesssim v(x)v(y)(\lambda|x - y|)^4 \log(\lambda|x - y|) \lesssim v(x)v(y)(\lambda|x - y|)^{4-}.$$

For large z, using the expansion of the Hankel function about $z = \infty$, recall (8), we have $|H_0^\pm(z)| \lesssim 1$ and $|\frac{d}{dz} H_0^\pm(z)| \lesssim z^{-1/2}$. So that for large $z > 1$, for $M_0^\pm(z)$ the $\log z$ term dominates and for $M_1^\pm(z)$ the $z^2 \log z$ term in (15) dominates, and we have

$$|M_0^\pm(\lambda)(x,y)\chi_{\{\lambda|x-y|>1\}}| \lesssim v(x)v(y)\log(\lambda|x-y|)\chi_{\{\lambda|x-y|>1\}} \lesssim v(x)v(y)(\lambda|x-y|)^{0+}\chi_{\{\lambda|x-y|>1\}},$$

$$|M_1^\pm(\lambda)(x,y)\chi_{\{\lambda|x-y|>1\}}| \lesssim v(x)v(y)(\lambda|x-y|)^{2+}\chi_{\{\lambda|x-y|>1\}}.$$

Hence, for any $0 < k < 2$, and for any $2 < \ell < 4$ we have

$$|M_0^\pm(\lambda)(x,y)| \lesssim v(x)v(y)\left[(\lambda|x-y|)^{2-}\chi_{\{\lambda|x-y|<1\}} + (\lambda|x-y|)^{0+}\chi_{\{\lambda|x-y|>1\}}\right] \lesssim v(x)v(y)(\lambda|x-y|)^k,$$

$$|M_1^\pm(\lambda)(x,y)| \lesssim v(x)v(y)\left[(\lambda|x-y|)^{4-}\chi_{\{\lambda|x-y|<1\}} + (\lambda|x-y|)^{2+}\chi_{\{\lambda|x-y|>1\}}\right] \lesssim v(x)v(y)(\lambda|x-y|)^\ell.$$

This yields the claim for M_0 and M_1 since $v(x)v(y)|x-y|^\ell$ is Hilbert-Schmidt from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ for $\beta > 1 + \ell$. For λ-derivatives, we note that

$$(18) \qquad |\partial_\lambda R_0^\pm(\lambda^2)(x,y)| \lesssim \left(\frac{|x-y|}{\lambda}\right)^{\frac{1}{2}},$$

and

$$\partial_\lambda F(\lambda|x-y|) = |x-y|\partial_z F(z)\Big|_{z=\lambda|x-y|}.$$

For the terms in M_0 and M_1 other than R_0, the effect of ∂_λ is comparable to division by λ. However, due to (18), on $\lambda|x-y| > 1$ we have for any $k \geq \frac{1}{2}$,

$$|\partial_\lambda M_0(\lambda)(x,y)| \lesssim v(x)v(y)\left[\left(\frac{|x-y|}{\lambda}\right)^{\frac{1}{2}} + \lambda^{-1}\right] \lesssim v(x)v(y)\lambda^{k-1}|x-y|^k.$$

Similarly,

$$|\partial_\lambda M_1(\lambda)(x,y)| \lesssim v(x)v(y)\left[\left(\frac{|x-y|}{\lambda}\right)^{\frac{1}{2}} + \lambda^{-1}(\lambda|x-y|)^\ell\right] \lesssim v(x)v(y)\lambda^{\ell-1}|x-y|^\ell. \qquad \square$$

We now give the definition of resonances from [14], also see [23]. Recall that $Q := \mathbb{1} - P$.

Definition 2.3.   (1) *We say zero is a regular point of the spectrum of $H = -\Delta + V$ provided $QTQ = Q(U + vG_0v)Q$ is invertible on $QL^2(\mathbb{R}^2)$.*

   (2) *Assume that zero is not a regular point of the spectrum. Let S_1 be the Riesz projection onto the kernel of QTQ as an operator on $QL^2(\mathbb{R}^2)$. Then $QTQ + S_1$ is invertible on $QL^2(\mathbb{R}^2)$. Accordingly, we define $D_0 = (QTQ + S_1)^{-1}$ as an operator on $QL^2(\mathbb{R}^2)$. We say there is a resonance of the first kind at zero if the operator $T_1 := S_1 T P T S_1$ is invertible on $S_1 L^2(\mathbb{R}^2)$.*

   (3) *We say there is a resonance of the second kind at zero if T_1 is not invertible on $S_1 L^2(\mathbb{R}^2)$ but $T_2 := S_2 v G_1 v S_2$ is invertible on $S_2 L^2(\mathbb{R}^2)$, where S_2 is the Riesz projection onto the kernel of T_1 (recall the definition of G_1 and G_2 in (12) and (13)).*

(4) *Finally, if T_2 is not invertible on $S_2 L^2(\mathbb{R}^2)$, we say there is a resonance of the third kind at zero. We note that in this case the operator $T_3 := S_3 v G_2 v S_3$ is always invertible on $S_3 L^2$, where S_3 is the Riesz projection onto the kernel of T_2 (see (6.41) in [14] or Section 5 below).*

Remarks. i) In [14], it is noted that the projections $S_1 - S_2$, $S_2 - S_3$ and S_3 correspond to s-wave resonances, p-wave resonances, and zero eigenspace respectively. In particular, resonance of the first kind means that there is only an s-wave resonance at zero. Resonance of the second kind means that there is a p-wave resonance, and there may or may not be an s-wave resonance. Finally, resonance of the third kind means that zero is an eigenvalue, and there may or may not be s-wave and p-wave resonances. We characterize these projections in Section 5.

ii) Since QTQ is self-adjoint, S_1 is the orthogonal projection onto the kernel of QTQ, and we have (with $D_0 = (QTQ + S_1)^{-1}$)

$$S_1 D_0 = D_0 S_1 = S_1.$$

This statement also valid for S_2 and $(T_1 + S_2)^{-1}$, and for S_3 and $(T_2 + S_3)^{-1}$.

iii) The operator QD_0Q is absolutely bounded in L^2. This was proved in Lemma 8 of [23] in the case $S_1 = 0$. With minor modifications, the same proof works in our case, too.

iv) The operators with kernel vG_iv are Hilbert-Schmidt operators on $L^2(\mathbb{R}^2)$ if $v(x) \lesssim \langle x \rangle^{-\beta}$ for $\beta > \frac{3}{2}$ if $i = 1$ and $\beta > 3$ for $i = 2, 3$.

To invert $M^{\pm}(\lambda) = U + vR_0^{\pm}(\lambda^2)v$, for small λ, we will use the following lemma (see Lemma 2.1 in [14]) repeatedly.

Lemma 2.4. *Let A be a closed operator on a Hilbert space $\mathcal{H}$ and S a projection. Suppose $A + S$ has a bounded inverse. Then A has a bounded inverse if and only if*

$$B := S - S(A + S)^{-1}S$$

has a bounded inverse in $S\mathcal{H}$, and in this case

$$A^{-1} = (A + S)^{-1} + (A + S)^{-1}SB^{-1}S(A + S)^{-1}.$$

We will apply this lemma with $A = M^{\pm}(\lambda)$ and $S = S_1$. Thus, we need to show that $M^{\pm}(\lambda) + S_1$ has a bounded inverse in $L^2(\mathbb{R}^2)$ and

(19) $$B_{\pm} = S_1 - S_1(M^{\pm}(\lambda) + S_1)^{-1}S_1$$

has a bounded inverse in $S_1 L^2(\mathbb{R}^2)$. We prove these claims and obtain expansions for the inverses for each type of resonance in Lemma 2.5, Proposition 2.6, and Proposition 4.1 below.

Lemma 2.5. *Suppose that zero is not a regular point of the spectrum of $H = -\Delta + V$, and let S_1 be the corresponding Riesz projection. Then for sufficiently small $\lambda_1 > 0$, the operators $M^{\pm}(\lambda) + S_1$ are invertible for all $0 < \lambda < \lambda_1$ as bounded operators on $L^2(\mathbb{R}^2)$. Further, one has*

$$(20) \qquad = h_{\pm}(\lambda)^{-1} S + Q D_0 Q + \mathcal{O}_1(\lambda^k),$$

for any $\frac{1}{2} \leq k < 2$ if $v(x) \lesssim \langle x \rangle^{-(1+k)-}$. Here $h_+(\lambda) = \overline{h_-(\lambda)} = a \ln \lambda + z$ (with $a \in \mathbb{R} \backslash \{0\}$ and $z \in \mathbb{C}$, $\Im z \neq 0$), and

$$(21) \qquad S = \begin{bmatrix} P & -PTQD_0Q \\ -QD_0QTP & QD_0QTPTQD_0Q \end{bmatrix}$$

is a finite-rank operator with real-valued kernel.

Proof. We will give the proof for M^+ and drop the superscript "+" from formulas. Using Lemma 2.2, we write $M(\lambda) + S_1$ with respect to the decomposition $L^2(\mathbb{R}^2) = PL^2(\mathbb{R}^2) \oplus QL^2(\mathbb{R}^2)$.

$$M(\lambda) + S_1 = \begin{bmatrix} g(\lambda)P + P(T + S_1)P & P(T + S_1)Q \\ Q(T + S_1)P & Q(T + S_1)Q \end{bmatrix} + M_0(\lambda).$$

Noting that $Q \geq S_1$, we have $S_1 P = PS_1 = 0$. Therefore,

$$M(\lambda) + S_1 = \begin{bmatrix} g(\lambda)P + PTP & PTQ \\ QTP & Q(T + S_1)Q \end{bmatrix} + M_0(\lambda).$$

Denote the matrix component of the above equation by $A(\lambda) = \{a_{ij}(\lambda)\}_{i,j=1}^2$.

Since $Q(T + S_1)Q$ is invertible, by the Fehsbach formula invertibility of $A(\lambda)$ hinges upon the existence of $d = (a_{11} - a_{12}a_{22}^{-1}a_{21})^{-1}$. Denoting $D_0 = (Q(T + S_1)Q)^{-1} : QL^2 \to QL^2$, we have

$$d = (g(\lambda)P + PTP - PTQD_0QTP)^{-1} = h(\lambda)^{-1}P$$

with $h(\lambda) = g(\lambda) + Tr(PTP - PTQD_0QTP) = a\ln(\lambda) + z$, with $a \in \mathbb{R}$ and $z \in \mathbb{C}$. This follows from (17) and the fact that $Tr(PTP - PTQD_0QTP)$ is λ independent and real-valued, as the kernels of T, QD_0Q and v are real-valued. Therefore, d exists if λ is sufficiently small.

Thus, by the Fehsbach formula,

$$A(\lambda)^{-1} = \begin{bmatrix} d & -da_{12}a_{22}^{-1} \\ -a_{22}^{-1}a_{21}d & a_{22}^{-1}a_{21}da_{12}a_{22}^{-1} + a_{22}^{-1} \end{bmatrix}$$

$$= h^{-1}(\lambda) \begin{bmatrix} P & -PTQD_0Q \\ -QD_0QTP & QD_0QTPTQD_0Q \end{bmatrix} + QD_0Q$$

$$(22) \qquad =: h^{-1}(\lambda)S + QD_0Q.$$

Note that S has rank at most two. This and the absolute boundedness of QD_0Q imply that $A^{-1}(\lambda) = \mathcal{O}_1(1)$.

Finally, we write

$$M(\lambda) + S_1 = A(\lambda) + M_0(\lambda) = [\mathbb{1} + M_0(\lambda)A^{-1}(\lambda)]A(\lambda).$$

Since $A^{-1}(\lambda) = \mathcal{O}_1(1)$ and, by Lemma 2.2, $M_0(\lambda) = \mathcal{O}_1(\lambda^k)$ provided $|v(x)| \lesssim \langle x \rangle^{-(1+k)-}$, we obtain

$$(23) \qquad (M^{\pm}(\lambda) + S_1)^{-1} = A_{\pm}^{-1}(\lambda)\big[\mathbb{1} + M_0^{\pm}(\lambda)A_{\pm}^{-1}(\lambda)\big]^{-1} = h_{\pm}(\lambda)^{-1}S + QD_0Q + \mathcal{O}_1(\lambda^k),$$

by a Neumann series expansion. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now prove the invertibility of the operators $B_{\pm} = S_1 - S_1(M^{\pm}(\lambda) + S_1)^{-1}S_1$.

Proposition 2.6. *Assume that $|v(x)| \lesssim \langle x \rangle^{-1-k-}$ for some $k \in [\frac{1}{2}, 2)$. Then, in the case of a resonance of the first kind, $B_{\pm}$ is invertible on $S_1 L^2(\mathbb{R}^2)$ and we have*

$$(24) \qquad\qquad\qquad\qquad\qquad B_{\pm}^{-1} = -h_{\pm}(\lambda)D_1 + \mathcal{O}_1(\lambda^k),$$

where $D_1 = T_1^{-1} = (S_1 TPTS_1)^{-1}$, and $h_{\pm}(\lambda)$ is as in Lemma 2.5.

Proof. We again prove the case of the "+" superscripts and subscripts and omit them from the notation. Using Lemma 2.5, we obtain

$$B = S_1 - S_1(h(\lambda)^{-1}S + QD_0Q)S_1 + \mathcal{O}_1(\lambda^{k+}).$$

Recall that $S_1 D_0 = D_0 S_1 = S_1$. Further, from the definition (21) of S, and the fact that $S_1 P = PS_1 = 0$, we obtain $S_1 S S_1 = S_1 TPTS_1 = T_1$. Therefore

$$(25) \qquad\qquad\qquad B = -h(\lambda)^{-1}S_1 SS_1 + \mathcal{O}_1(\lambda^{k+}) = -h(\lambda)^{-1}T_1 + \mathcal{O}_1(\lambda^{k+}).$$

Recall that by the definition of a resonance of the first kind, the leading term T_1 in the definition of B is invertible on $S_1 L^2(\mathbb{R}^2)$. Therefore, for sufficiently small λ,

$$B^{-1} = -h(\lambda)[T_1 - h(\lambda)\mathcal{O}_1(\lambda^{k+})]^{-1} = -h(\lambda)[T_1 + \mathcal{O}_1(\lambda^{k+})]^{-1} = -h(\lambda)D_1 + \mathcal{O}_1(\lambda^k).$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$$

Combining Lemma 2.4, Lemma 2.5, and Proposition 2.6, we obtain

Corollary 2.7. *Assume that $|v(x)| \lesssim \langle x \rangle^{-1-k-}$ for some $k \in [\frac{1}{2}, 2)$. Then in the case of a resonance of the first kind, we have*

$$M^{\pm}(\lambda)^{-1} = -h_{\pm}(\lambda)S_1 D_1 S_1 - SS_1 D_1 S_1 - S_1 D_1 S_1 S$$

$$- h_{\pm}(\lambda)^{-1}SS_1 D_1 S_1 S + h_{\pm}(\lambda)^{-1}S + QD_0Q + \mathcal{O}_1(\lambda^k),$$

provided that λ is sufficiently small.

Proof. Combining Lemma 2.4, Lemma 2.5, and Proposition 2.6, we have

$$M^{\pm}(\lambda)^{-1} = (M^{\pm}(\lambda) + S_1)^{-1} + (M^{\pm}(\lambda) + S_1)^{-1} S_1 B^{-1} S_1 (M^{\pm}(\lambda) + S_1)^{-1}$$

$$= h_{\pm}(\lambda)^{-1} S + Q D_0 Q - h(\lambda)\big(h_{\pm}(\lambda)^{-1} S + Q D_0 Q\big) S_1 D_1 S_1 \big(h_{\pm}(\lambda)^{-1} S + Q D_0 Q\big) + \mathcal{O}_1(\lambda^k)$$

$$= -h_{\pm}(\lambda) S_1 D_1 S_1 - S S_1 D_1 S_1 - S_1 D_1 S_1 S - h_{\pm}(\lambda)^{-1} S S_1 D_1 S_1 S$$

$$+ h_{\pm}(\lambda)^{-1} S + Q D_0 Q + \mathcal{O}_1(\lambda^k).$$

Here we used the fact that $S_1 Q D_0 Q = Q D_0 Q S_1 = S_1$. $\qquad\square$

Remark. Under the conditions of Corollary 2.7, the resolvent identity

(26) $$R_V^{\pm}(\lambda^2) = R_0^{\pm}(\lambda^2) - R_0^{\pm}(\lambda^2) v M^{\pm}(\lambda)^{-1} v R_0^{\pm}(\lambda^2)$$

holds as an operator identity on $L^{2,\frac{1}{2}+}(\mathbb{R}^2) \to L^{2,-\frac{1}{2}-}(\mathbb{R}^2)$, as in the limiting absorption principle, [2].

3. Resonance of the first kind

In this section, we establish the estimates needed to prove Theorem 1.1. We assume that there is a resonance of the first kind, λ_1 is sufficiently small (so that the analysis in the previous section is valid), and that $v(x) \lesssim \langle x \rangle^{-(1+k)-}$ for $k = 1$, or equivalently $|V(x)| \lesssim \langle x \rangle^{-4-}$. It suffices to prove that

Theorem 3.1. *Under the conditions above, we have*

(27) $$|\langle K_{\lambda_1} f, g \rangle| \lesssim |t|^{-1},$$

for Schwartz functions f and g with $\|f\|_1 = \|g\|_1 = 1$.

This theorem will be established in Propositions 3.2, 3.11, 3.12, and 3.13. All statements in this section are valid under the conditions above.

Proposition 3.2. *The contribution of the first term in Corollary 2.7 in (1) satisfies (27). More explicitly, we have the bound*

$$\left| \int_{\mathbb{R}^8} \int_0^{\infty} e^{it\lambda^2} \lambda \chi(\lambda) \mathcal{K}(\lambda, p, q) v(x_1) S_1 D_1 S_1(x_1, y_1) v(y_1) f(x) g(y) d\lambda dx_1 dy_1 dx dy \right| \lesssim |t|^{-1},$$

where $p = |x - x_1|$, $q = |y - y_1|$, and

(28) $$\mathcal{K}(\lambda, p, q) = h^+(\lambda) H_0^+(\lambda p) H_0^+(\lambda q) - h^-(\lambda) H_0^-(\lambda p) H_0^-(\lambda q)$$

$$= 2ia \log(\lambda)[Y_0(\lambda p) J_0(\lambda q) + J_0(\lambda p) Y_0(\lambda q)] + 2z[J_0(\lambda p) J_0(\lambda q) + Y_0(\lambda p) Y_0(\lambda q)].$$

To prove this proposition, we need to consider the high and low energy contributions of the Bessel functions separately. To this end we use the partitions of unity $1 = \chi(\lambda|y - y_1|) + \tilde{\chi}(\lambda|y - y_1|)$ and $1 = \chi(\lambda|x - x_1|) + \tilde{\chi}(\lambda|x - x_1|)$. We divide the proof of Proposition 3.2 into Lemmas 3.4, 3.8, 3.10 and their respective corollaries, Corollaries 3.5, 3.9, due to the various terms arising in (28).

For the low energy parts, the following lemma will be useful:

Lemma 3.3. *Let $p = |x - x_1|$, $q = |x| + 1$, and*

$$F(\lambda, x, x_1) := \chi(\lambda p)Y_0(\lambda p) - \chi(\lambda q)Y_0(\lambda q),$$

$$G(\lambda, x, x_1) := \chi(\lambda p)J_0(\lambda p) - \chi(\lambda q)J_0(\lambda q).$$

Then for any $\tau \in [0,1]$ and $\lambda \leq 2\lambda_1$ we have

$$|G(\lambda, x, x_1)| \lesssim \lambda^\tau \langle x_1 \rangle^\tau, \qquad |\partial_\lambda G(\lambda, x, x_1)| \lesssim \langle x_1 \rangle^\tau \lambda^{\tau-1},$$

$$|F(\lambda, x, x_1)| \leq \int_0^{2\lambda_1} |\partial_\lambda F(\lambda, x, x_1)|d\lambda + |F(0+, x, x_1)| \lesssim k(x, x_1), \qquad |\partial_\lambda F(\lambda, x, x_1)| \lesssim \frac{1}{\lambda}.$$

Here $k(x, x_1) := 1 + \log^+ |x_1| + \log^- |x - x_1|$, where $\log^- y := \chi_{\{0 < y < 1\}} |\log y|$ and $\log^+ y := \chi_{\{y > 1\}} \log y$. G

Proof. We start with G. Let $g(s) := \chi(s)J_0(s)$. We have $g'(s) = O(1)$. Therefore, by the mean value theorem and the boundedness of g, we have

$$|G(\lambda, x, x_1)| \lesssim \min(\lambda|p - q|, 1) \lesssim \min(\lambda\langle x_1 \rangle, 1) \lesssim \lambda^\tau \langle x_1 \rangle^\tau,$$

for any $0 \leq \tau \leq 1$.

Now consider

$$|\partial_\lambda G(\lambda, x, x_1)| = |pg'(\lambda p) - qg'(\lambda q)|.$$

Let $g_1(s) = sg'(s)$. We have $|g_1(s)| \lesssim 1$ and $|g_1'(s)| \lesssim 1$. Therefore, by the mean value theorem and the boundedness of g_1, we have

$$|\partial_\lambda G(\lambda, x, x_1)| = \left| \frac{g_1(\lambda p) - g_1(\lambda q)}{\lambda} \right| \lesssim \left| \frac{\min(\lambda|p - q|, 1)}{\lambda} \right| \lesssim \langle x_1 \rangle^\tau \lambda^{\tau-1},$$

for any $\tau \in [0,1]$.

The bounds for F were obtained in [23]. We repeat them for completeness. Note that $F(0+, x, x_1) = \log\left(\frac{|x - x_1|}{|x| + 1}\right) + c \lesssim k(x, x_1)$. Therefore it suffices to bound

$$(29) \qquad \int_0^{2\lambda_1} |\partial_\lambda F(\lambda, x, x_1)|d\lambda \lesssim \int_0^{2\lambda_1} p|\chi'(\lambda p) \log(\lambda p)|d\lambda + \int_0^{2\lambda_1} q|\chi'(\lambda q) \log(\lambda q)|d\lambda$$

$$(30) \qquad \qquad + \int_0^{2\lambda_1} \frac{1}{\lambda}|\chi(\lambda p) - \chi(\lambda q)|d\lambda.$$

By inspecting the integrands on the right hand side, we see that $|\partial_\lambda F|$ is bounded by $1/\lambda$. To obtain the statement for $|F|$ first note that, since χ' is supported in the set $[\lambda_1/2, 2\lambda_1]$, the first line in (29) is $\lesssim 1$. To estimate the second line note that $\chi(\lambda p) - \chi(\lambda q)$ is supported on the set $[\frac{\lambda_1}{2p}, 2\frac{\lambda_1}{q}]$, which implies that the last line is $\lesssim \left| \log\left(\frac{|x-x_1|}{|x|+1} \right) \right| \lesssim k(x, x_1)$. $\qquad\square$

Lemma 3.4. *We have the bound*

$$(31) \quad \left| \int_{\mathbb{R}^8} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \log \lambda \chi(\lambda|x-x_1|) Y_0(\lambda|x-x_1|) v(x_1) S_1 D_1 S_1(x_1, y_1) v(y_1) \right.$$

$$\left. J_0(\lambda|y-y_1|)\chi(\lambda|y-y_1|) \, d\lambda f(x) g(y) \, dx_1 \, dy_1 \, dx \, dy \right| \lesssim |t|^{-1}.$$

Proof. Since $S_1 \leq Q$ are projections and Q is the projection orthogonal to v, we have

$$(32) \quad \int_{\mathbb{R}^4} v(x) [S_1 D_1 S_1](x, y) h(y) dx \, dy = \int_{\mathbb{R}^4} h(x) [S_1 D_1 S_1](x, y) v(y) \, dx \, dy = 0$$

for all $h \in L^2(\mathbb{R}^2)$. As such, we can subtract functions of x (resp. y) only from χY_0 (resp. χJ_0) in the integrand of (31). We use the functions defined in Lemma 3.3. Thus we replace $\chi(\lambda|x-x_1|) Y_0(\lambda|x-x_1|)$ with $F(\lambda, x, x_1)$ and $\chi(\lambda|y-y_1|) J_0(\lambda|y-y_1|)$ with $G(\lambda, y, y_1)$ on the left hand side of (31). Therefore the λ integral of (31) is equivalent to

$$(33) \quad \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \log(\lambda) F(\lambda, x, x_1) G(\lambda, y, y_1) \, d\lambda.$$

We integrate by parts once to get

$$(34) \quad (33) \lesssim |t|^{-1} \int_0^\infty [\log \lambda \chi'(\lambda) + \lambda^{-1}] |F(\lambda, x, x_1)| |G(\lambda, y, y_1)| \, d\lambda$$

$$(35) \quad + |t|^{-1} \int_0^\infty |\chi(\lambda) \log \lambda| |\partial_\lambda F(\lambda, x, x_1)| |G(\lambda, y, y_1)| \, d\lambda$$

$$(36) \quad + |t|^{-1} \int_0^\infty |\chi(\lambda) \log \lambda| |F(\lambda, x, x_1)| |\partial_\lambda G(\lambda, y, y_1)| \, d\lambda.$$

There is no boundary term since, by Lemma 3.3, we have that $F(0+, y_1, y) \lesssim k(x, x_1)$ and $G(0, y, y_1) = 0$. From Lemma 3.3 again, we have for any $\tau \in (0, 1]$

$$(34) \lesssim \int_0^{2\lambda_1} [\log \lambda + \lambda^{-1}] k(x, x_1) \lambda^\tau \langle y_1 \rangle^\tau \, d\lambda \lesssim \langle y_1 \rangle^\tau k(x, x_1).$$

Taking $\tau = 0+$, this term now contributes the following to (31),

$$(37) \quad \lesssim |t|^{-1} \int_{\mathbb{R}^8} k(x, x_1) v(x_1) |D_1(x_1, y_1)| v(y_1) \langle y_1 \rangle^{0+} |f(x)| |g(y)| \, dx_1 \, dy_1 \, dx \, dy$$

$$\lesssim |t|^{-1} \sup_{x \in \mathbb{R}^2} \|k(x, \cdot) v(\cdot)\|_2 \|D_1\|_{2 \to 2} \|v\|_{L^{2,0+}} \|f\|_1 \|g\|_1 \lesssim |t|^{-1}.$$

For the case of (35) and (36), we again note the bounds in Lemma 3.3, and that on the support of $\chi(\lambda)$, $|\lambda^\tau \log \lambda| \lesssim 1$ for any $\tau > 0$. The desired bound follows as in (37). $\qquad\square$

We also need the following bounds taking care of the contributions of the remaining terms in (28):

Corollary 3.5. *For $\mathcal{C}(z) = J_0(z)$ or $\mathcal{C}(z) = Y_0(z)$, we have the bound*

$$(38) \quad \left| \int_{\mathbb{R}^8} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \chi(\lambda|x - x_1|) \mathcal{C}(\lambda|x - x_1|) v(x_1) S_1 D_1 S_1(x_1, y_1) v(y_1) \right.$$

$$\left. \mathcal{C}(\lambda|y - y_1|) \chi(\lambda|y - y_1|) \, d\lambda f(x) g(y) \, dx_1 \, dy_1 \, dx \, dy \right| \lesssim |t|^{-1}.$$

Proof. Using the notation of Lemma 3.4, we need to bound

$$\left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) F(\lambda, x, x_1) F(\lambda, y, y_1) \, d\lambda \right|,$$

and the similar term when F is replaced by G. This follows easily from one integration by parts and the bounds of Lemma 3.3 as in the previous lemma. $\qquad\square$

We now need to bound the resulting terms when one of the Bessel functions is supported on large energies. The following variation of stationary phase from [23] will be useful in the analysis. For completeness we give the proof.

Lemma 3.6. *Let $\phi'(0) = 0$ and $1 \le \phi'' \le C$. Then,*

$$\left| \int_{-\infty}^\infty e^{it\phi(\lambda)} a(\lambda) \, d\lambda \right| \lesssim \int_{|\lambda| < |t|^{-\frac{1}{2}}} |a(\lambda)| \, d\lambda + |t|^{-1} \int_{|\lambda| > |t|^{-\frac{1}{2}}} \left(\frac{|a(\lambda)|}{|\lambda^2|} + \frac{|a'(\lambda)|}{|\lambda|} \right) d\lambda.$$

Proof. Let $\eta \in C_c^\infty(\mathbb{R})$ be such that $\eta(x) = 1$ if $|x| < 1$ and $\eta(x) = 0$ if $|x| > 2$. Let $\eta_2(x) = \eta(x/2|t|^{-1/2})$. Writing $1 = \eta_2 + (1 - \eta_2)$, we rewrite the integral as follows

$$\left| \int_{-\infty}^\infty e^{it\phi(\lambda)} a(\lambda) \, d\lambda \right| \lesssim \left| \int_{-\infty}^\infty e^{it\phi(\lambda)} a(\lambda) \eta_2(\lambda) \, d\lambda \right| + \left| \int_{-\infty}^\infty e^{it\phi(\lambda)} a(\lambda)(1 - \eta_2(\lambda)) \, d\lambda \right|$$

The first term is bounded as in the claim since $\text{supp}(\eta_2) = [-|t|^{-\frac{1}{2}}, |t|^{-\frac{1}{2}}]$. For the second term, we integrate by parts once in λ to bound with

$$|t|^{-1} \left| \int_{-\infty}^\infty e^{it\phi(\lambda)} \frac{d}{d\lambda} \left(\frac{a(\lambda)(1 - \eta_2(\lambda))}{\phi'(\lambda)} \right) d\lambda \right|$$

By Taylor's Theorem,

$$\phi'(\lambda) = \phi'(0) + \lambda \phi''(c) = \lambda \phi''(c) \approx \lambda,$$

Considering the terms when the derivative acts on $a(\lambda)$, $1 - \eta_2(\lambda)$ and $1/\phi'(\lambda)$ finishes the proof. $\quad\square$

In addition we have the following high-energy analogue of Lemma 3.3. In light of the high energy representations of the Bessel functions (9), recall that for $\mathcal{C} \in \{J_0, Y_0, H_0\}$,

$$\mathcal{C}(y)\widetilde{\chi}(y) = e^{iy}\omega_+(y) + e^{-iy}\omega_-(y), \qquad |\omega_\pm^{(\ell)}(y)| \lesssim \langle y \rangle^{-\frac{1}{2} - \ell}.$$

Lemma 3.7. *Define for $p, q > 0$*

$$(39) \qquad \widetilde{G}^{\pm}(\lambda, p, q) := \widetilde{\chi}(\lambda p)\omega_{\pm}(\lambda p) - e^{\pm i\lambda(p-q)}\widetilde{\chi}(\lambda q)\omega_{\pm}(\lambda q).$$

with $\omega_{\pm}$ as in (9). Then for any $0 \leq \tau \leq 1$ and $\lambda \leq 2\lambda_1$,

$$|\widetilde{G}^{\pm}(\lambda, p, q)| \lesssim (\lambda|p - q|)^{\tau}\left(\frac{\widetilde{\chi}(\lambda p)}{|\lambda p|^{\frac{1}{2}}} + \frac{\widetilde{\chi}(\lambda q)}{|\lambda q|^{\frac{1}{2}}}\right),$$

$$|\partial_{\lambda}\widetilde{G}^{\pm}(\lambda, p, q)| \lesssim |p - q|\left(\frac{\widetilde{\chi}(\lambda p)}{|\lambda p|^{\frac{1}{2}}} + \frac{\widetilde{\chi}(\lambda q)}{|\lambda q|^{\frac{1}{2}}}\right)$$

Proof. We note first that from (9), we have

$$(40) \qquad |\widetilde{G}^{\pm}(\lambda, p, q)| \lesssim \frac{\widetilde{\chi}(\lambda p)}{|\lambda p|^{\frac{1}{2}}} + \frac{\widetilde{\chi}(\lambda q)}{|\lambda q|^{\frac{1}{2}}}.$$

We consider the case of $\widetilde{G}^{+}$, the case of $\widetilde{G}^{-}$ is similar. Define the function

$$b(s) := \widetilde{\chi}(s)\omega_{+}(s).$$

Using (9), one obtains that for $k = 0, 1, 2, ...$,

$$|b^{(k)}(s)| \lesssim \widetilde{\chi}(s)|s|^{-\frac{1}{2}-k}.$$

We now rewrite $\widetilde{G}$ in terms of b:

$$\widetilde{G}(\lambda, p, q) = b(\lambda p) - b(\lambda q) + \left(1 - e^{i\lambda(p-q)}\right)b(\lambda q).$$

Note that the absolute value of the last summand is

$$\lesssim \lambda|p - q|\frac{\widetilde{\chi}(\lambda q)}{|\lambda q|^{\frac{1}{2}}}.$$

To estimate the difference of the first two we assume without loss of generality that $p > q$ and write

$$|b(\lambda p) - b(\lambda q)| = \left|\int_{\lambda q}^{\lambda p} b'(s)ds\right| \lesssim \int_{\lambda q}^{\lambda p} \widetilde{\chi}(s)|s|^{-\frac{3}{2}}ds.$$

In the case, $1 < \lambda q < \lambda p$, we estimate this integral by

$$\lambda|p - q|\frac{\widetilde{\chi}(\lambda q)}{|\lambda q|^{\frac{3}{2}}}.$$

In the case $\lambda q < 1 < \lambda p$, we estimate it as follows

$$\int_{\lambda q}^{\lambda p} \widetilde{\chi}(s)|s|^{-\frac{3}{2}}ds \lesssim \widetilde{\chi}(\lambda p)\int_{1}^{\lambda p} s^{-3/2}ds \lesssim \widetilde{\chi}(\lambda p)\frac{(\lambda p)^{1/2} - 1}{|\lambda p|^{1/2}} \leq$$

$$\widetilde{\chi}(\lambda p)\frac{(\lambda p)^{1/2} - (\lambda q)^{1/2}}{|\lambda p|^{1/2}} \lesssim \lambda|p - q|\frac{\widetilde{\chi}(\lambda p)}{|\lambda p|}.$$

Combining these bounds and interpolating with (40) we obtain the first assertion of the lemma.

We now turn to the derivative. We note that

$$\partial_\lambda \widetilde{G}(\lambda, p, q) = pb'(\lambda p) - qb'(\lambda q) + \big(1 - e^{i\lambda(p-q)}\big)qb'(\lambda q) - i(p-q)e^{i\lambda(p-q)}b(\lambda q)$$

$$= \frac{1}{\lambda}[b_1(\lambda p) - b_1(\lambda q)] + \frac{1 - e^{i\lambda(p-q)}}{\lambda}b_1(\lambda q) - i(p-q)e^{i\lambda(p-q)}b(\lambda q),$$

where $b_1(s) := sb'(s)$ satisfies the same bounds that $b(s)$ does. Therefore the second assertion of the lemma follows as above. $\qquad\square$

Lemma 3.8. *We have the bound*

$$(41) \quad \left| \int_{\mathbb{R}^8} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \log \lambda \chi(\lambda|x - x_1|) Y_0(\lambda|x - x_1|) v(x_1) S_1 D_1 S_1(x_1, y_1) v(y_1) \right.$$

$$\left. J_0(\lambda|y - y_1|)\widetilde{\chi}(\lambda|y - y_1|) \, d\lambda f(x)g(y) \, dx_1 \, dy_1 \, dx \, dy \right| \lesssim |t|^{-1}.$$

Proof. Without loss of generality we assume that $t > 0$. As in the proof of the previous statements, it suffices to prove that for fixed x, x_1, y, y_1 the λ-integral is bounded by $k(x, x_1)\langle y_1 \rangle t^{-1}$. This power of $\langle y_1 \rangle$ necessitates extra decay on the potential to push through the L^2 mapping bounds as in the previous lemmas. Accordingly, we assume that $v(x) \lesssim \langle x \rangle^{-2-}$ or equivalently that $|V(x)| \lesssim \langle x \rangle^{-4-}$.

Let $p = \max(|y - y_1|, 1 + |y|)$ and $q = \min(|y - y_1|, 1 + |y|)$. Using (32), it suffices to consider

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \log \lambda \, F(\lambda, x, x_1)(J_0(\lambda p)\widetilde{\chi}(\lambda p) - J_0(\lambda q)\widetilde{\chi}(\lambda q)) \, d\lambda,$$

where $F(\lambda, x, x_1)$ is as in Lemma 3.3. The oscillatory term in the definition (9) of J_0 for large energies will move the stationary point of the phase. Pulling out the slower oscillation $e^{\pm i\lambda q}$, we rewrite this integral as a sum of

$$\int_0^\infty e^{it\phi_\pm(\lambda)} \lambda \chi(\lambda) \log \lambda \, F(\lambda, x, x_1)\widetilde{G}^\pm(\lambda, p, q) \, d\lambda,$$

where $\phi_\pm(\lambda) = \lambda^2 \pm \lambda q t^{-1}$, and $\widetilde{G}$ is from Lemma 3.7. Note that this moves the stationary point of the oscillatory integral to $\lambda_0 = \mp\frac{q}{2t} = \mp\frac{\min(|y-y_1|, 1+|y|)}{2t}$.

We first consider the contribution of the term with the phase $\phi_-(\lambda)$ in which case the critical point satisfies $\lambda_0 \geq 0$. Let

$$a(\lambda) := \lambda \chi(\lambda) \log \lambda F(\lambda, x, x_1)\widetilde{G}(\lambda, p, q).$$

Using the bounds in Lemma 3.3 and Lemma 3.7 (with $\tau = 0+$), we have

$$(42) \quad |a(\lambda)| \lesssim k(x, x_1)\langle y_1 \rangle^{0+} \lambda \chi(\lambda) \left(\frac{\widetilde{\chi}(\lambda p)}{|\lambda p|^{\frac{1}{2}}} + \frac{\widetilde{\chi}(\lambda q)}{|\lambda q|^{\frac{1}{2}}} \right), \quad \text{and}$$

$$(43) \quad |a'(\lambda)| \lesssim k(x, x_1)\langle y_1 \rangle \chi(\lambda) \left(\frac{\widetilde{\chi}(\lambda p)}{|\lambda p|^{\frac{1}{2}}} + \frac{\widetilde{\chi}(\lambda q)}{|\lambda q|^{\frac{1}{2}}} \right).$$

We now apply Lemma 3.6 with $a(\lambda)$ as above to bound the λ-integral in this case by

$$(44) \qquad \int_{|\lambda-\lambda_0|<t^{-1/2}} |a(\lambda)|\, d\lambda + t^{-1} \int_{|\lambda-\lambda_0|>t^{-1/2}} \left(\frac{|a(\lambda)|}{|\lambda-\lambda_0|^2} + \frac{|a'(\lambda)|}{|\lambda-\lambda_0|} \right) d\lambda.$$

Using (42), we bound the first integral in (44) by

$$(45) \qquad \int_{|\lambda-\lambda_0|<t^{-1/2}} |a(\lambda)|\, d\lambda \lesssim k(x,x_1)\langle y_1\rangle^{0+} \int_{|\lambda-\lambda_0|<t^{-\frac{1}{2}}} \sqrt{\lambda}\left(\frac{\widetilde{\chi}(\lambda p)}{p^{\frac{1}{2}}} + \frac{\widetilde{\chi}(\lambda q)}{q^{\frac{1}{2}}} \right) d\lambda.$$

There are two cases: $\lambda_0 \gtrsim t^{-\frac{1}{2}}$ and $\lambda_0 \lesssim t^{-\frac{1}{2}}$. In the former case, on the support of the integral, we have $\lambda \lesssim \lambda_0$. Therefore,

$$(45) \lesssim k(x,x_1)\langle y_1\rangle^{0+} t^{-\frac{1}{2}}\lambda_0^{\frac{1}{2}}(p^{-\frac{1}{2}}+q^{-\frac{1}{2}}) \lesssim t^{-1}k(x,x_1)\langle y_1\rangle^{0+}.$$

In the last inequality, we used $p^{-1}\lambda_0 \leq q^{-1}\lambda_0 \leq t^{-1}$. In the latter case, on the support of the integral, we have $\lambda \lesssim t^{-\frac{1}{2}}$. So that

$$(45) \lesssim k(x,x_1)\langle y_1\rangle^{0+} \int_0^{t^{-\frac{1}{2}}} \sqrt{\lambda}\left(\frac{\widetilde{\chi}(\lambda p)}{p^{\frac{1}{2}}} + \frac{\widetilde{\chi}(\lambda q)}{q^{\frac{1}{2}}} \right) d\lambda$$

For the $\widetilde{\chi}(\lambda p)$ term to have any contribution to the integral, we must have that $p^{-1} \lesssim t^{-\frac{1}{2}}$, similarly for q^{-1}. So that,

$$(45) \lesssim k(x,x_1)\langle y_1\rangle^{0+}(p^{-\frac{1}{2}}+q^{-\frac{1}{2}})t^{-\frac{3}{4}} \lesssim k(x,x_1)\langle y_1\rangle^{0+}t^{-1}.$$

It suffices to bound the second integral in (44) by $k(x,x_1)\langle y_1\rangle$. We first establish the bounds for the $a(\lambda)$ term and then consider the derivative $a'(\lambda)$.

We have two cases: $\lambda_0 \ll t^{-\frac{1}{2}}$ and $\lambda_0 \gtrsim t^{-\frac{1}{2}}$. In the former case, we have $|\lambda-\lambda_0| \approx \lambda$. Thus, using (42), we obtain

$$\int_{|\lambda-\lambda_0|>t^{-1/2}} \frac{|a(\lambda)|}{|\lambda-\lambda_0|^2}\, d\lambda \lesssim k(x,x_1)\langle y_1\rangle^{0+} \int_{\mathbb{R}} \lambda^{-\frac{3}{2}}\left(\frac{\widetilde{\chi}(\lambda p)}{p^{\frac{1}{2}}} + \frac{\widetilde{\chi}(\lambda q)}{q^{\frac{1}{2}}} \right) d\lambda \lesssim k(x,x_1)\langle y_1\rangle^{0+}.$$

In the latter case, we have

$$\int_{|\lambda-\lambda_0|>t^{-1/2}} \frac{|a(\lambda)|}{|\lambda-\lambda_0|^2}\, d\lambda \lesssim k(x,x_1)\langle y_1\rangle^{0+} \int_{|\lambda-\lambda_0|>t^{-\frac{1}{2}}} \frac{\lambda^{\frac{1}{2}}}{|\lambda-\lambda_0|^2}\left(\frac{\widetilde{\chi}(\lambda p)}{p^{\frac{1}{2}}} + \frac{\widetilde{\chi}(\lambda q)}{q^{\frac{1}{2}}} \right) d\lambda$$

Changing the variable $s = \lambda - \lambda_0$ and recalling that $p \geq q$, we bound this by

$$k(x,x_1)\langle y_1\rangle^{0+}q^{-\frac{1}{2}} \int_{|s|>t^{-\frac{1}{2}}} \frac{s^{\frac{1}{2}}+\lambda_0^{\frac{1}{2}}}{s^2}\, ds \lesssim k(x,x_1)\langle y_1\rangle^{0+}q^{-\frac{1}{2}}(t^{1/4}+t^{1/2}\lambda_0^{1/2}) \lesssim k(x,x_1)\langle y_1\rangle^{0+}.$$

The last inequality follows from the assumption $t^{-1/2} \lesssim \lambda_0 = \frac{q}{2t}$.

Now, we consider the contribution of $a'(\lambda)$. Again we have two cases: $\lambda_0 \ll t^{-\frac{1}{2}}$ and $\lambda_0 \gtrsim t^{-\frac{1}{2}}$. In the former case, we have $|\lambda-\lambda_0| \approx \lambda$. Thus, using (43), we obtain

$$\int_{|\lambda-\lambda_0|>t^{-1/2}} \frac{|a'(\lambda)|}{|\lambda-\lambda_0|}\, d\lambda \lesssim k(x,x_1)\langle y_1\rangle \int_{\mathbb{R}} \lambda^{-\frac{3}{2}}\left(\frac{\widetilde{\chi}(\lambda p)}{p^{\frac{1}{2}}} + \frac{\widetilde{\chi}(\lambda q)}{q^{\frac{1}{2}}} \right) d\lambda \lesssim k(x,x_1)\langle y_1\rangle.$$

In the latter case, we have

$$\int_{|\lambda-\lambda_0|>t^{-1/2}} \frac{|a'(\lambda)|}{|\lambda-\lambda_0|}\, d\lambda \lesssim k(x,x_1)\langle y_1\rangle \int_{|\lambda-\lambda_0|>t^{-\frac{1}{2}}} \frac{1}{|\lambda-\lambda_0|\lambda^{\frac{1}{2}}}\left(\frac{\widetilde{\chi}(\lambda p)}{p^{\frac{1}{2}}}+\frac{\widetilde{\chi}(\lambda q)}{q^{\frac{1}{2}}}\right)d\lambda$$

$$\lesssim k(x,x_1)\langle y_1\rangle\left[q^{-\frac{1}{2}}\int_{|\lambda-\lambda_0|>t^{-\frac{1}{2}}}\frac{d\lambda}{|\lambda-\lambda_0|^{\frac{3}{2}}}+\int_{\mathbb{R}}\frac{d\lambda}{\lambda^{\frac{3}{2}}}\left(\frac{\widetilde{\chi}(\lambda p)}{p^{\frac{1}{2}}}+\frac{\widetilde{\chi}(\lambda q)}{q^{\frac{1}{2}}}\right)\right]$$

$$\lesssim k(x,x_1)\langle y_1\rangle\left[q^{-\frac{1}{2}}t^{1/4}+1\right]\lesssim k(x,x_1)\langle y_1\rangle.$$

The second inequality follows from $\frac{1}{|\lambda-\lambda_0|\lambda^{1/2}}\leq\frac{1}{|\lambda-\lambda_0|^{3/2}}+\frac{1}{\lambda^{3/2}}$, and the last one from the assumption $t^{-1/2}\lesssim\lambda_0=\frac{q}{2t}$.

When considering the phase $\phi_+(\lambda)=\lambda^2+\lambda qt^{-1}$, integration by parts suffices to obtain the desired bound since the phase has no critical points on $(0,\infty)$. We have

$$\left|\int_0^\infty e^{it\phi_+(\lambda)}a(\lambda)\, d\lambda\right|\lesssim t^{-1}\left(\int_0^\infty\frac{|a(\lambda)|}{|\phi_+'(\lambda)|^2}\, d\lambda+\int_0^\infty\frac{|a'(\lambda)|}{|\phi_+'(\lambda)|}\, d\lambda\right).$$

Using (42), (43), and $\phi_+'(\lambda)\geq 2\lambda$, we bound the right hand side by

$$\lesssim k(x,x_1)\langle y_1\rangle t^{-1}\int_0^\infty\lambda^{-\frac{3}{2}}\left(\frac{\widetilde{\chi}(\lambda p)}{p^{\frac{1}{2}}}+\frac{\widetilde{\chi}(\lambda q)}{q^{\frac{1}{2}}}\right)d\lambda\lesssim k(x,x_1)\langle y_1\rangle t^{-1}. \qquad\square$$

When switching roles of $\widetilde{\chi}$ and χ in (41), we note that from (9) the high energy Bessel function representation holds for $Y_0(y)$ as well. The proof will move along the same line, with $G(\lambda,y,y_1)$ in place of $\widetilde{G}(\lambda,y,y_1)$ and using $\widetilde{G}(\lambda,x,x_1)$ in place of $F(\lambda,x,x_1)$. The case when both Bessel functions are Y_0 or J_0 is similar:

Corollary 3.9. *For $\mathcal{C}(z)=J_0(z)$ or $\mathcal{C}(z)=Y_0(z)$, we have the bound*

$$(46)\quad \left|\int_{\mathbb{R}^8}\int_0^\infty e^{it\lambda^2}\lambda\chi(\lambda)\chi(\lambda|x-x_1|)\mathcal{C}(\lambda|x-x_1|)v(x_1)S_1D_1S_1(x_1,y_1)v(y_1)\right.$$

$$\left.\mathcal{C}(\lambda|y-y_1|)\widetilde{\chi}(\lambda|y-y_1|)\, d\lambda f(x)g(y)\, dx_1\, dy_1\, dx\, dy\right|\lesssim |t|^{-1}.$$

We now consider the case when both Bessel functions are supported on high energies. For this we will use the first line of (28):

Lemma 3.10. *We have the bound*

$$(47)\quad \left|\int_{\mathbb{R}^8}\int_0^\infty e^{it\lambda^2}\lambda\chi(\lambda)h^\pm(\lambda)\widetilde{\chi}(\lambda|x-x_1|)H_0^\pm(\lambda|x-x_1|)v(x_1)S_1D_1S_1(x_1,y_1)v(y_1)\right.$$

$$\left.H_0^\pm(\lambda|y-y_1|)\widetilde{\chi}(\lambda|y-y_1|)\, d\lambda f(x)g(y)\, dx_1\, dy_1\, dx\, dy\right|\lesssim |t|^{-1}.$$

Proof. Again we assume that $t>0$. Recall (8):

$$(48)\qquad\qquad\qquad H_0^\pm(z)\widetilde{\chi}(z)=e^{\pm iz}\omega_\pm(z)$$

with $|\omega_\pm^{(\ell)}(z)| \lesssim (1+|z|)^{-\frac{1}{2}-\ell}$. As in Lemma 3.8 we need to use the auxiliary function $\widetilde{G}$. Denote $p_1 = \max(|x-x_1|, 1+|x|)$, $p_2 = \min(|x-x_1|, 1+|x|)$, $q_1 = \max(|y-y_1|, 1+|y|)$ and $q_2 = \min(|y-y_1|, 1+|y|)$. Without loss of generality, $p_1, p_2, q_1, q_2 > 0$. We note that by (32), we can replace $H_0^\pm(\lambda|x-x_1|)$ with $\widetilde{G}(\lambda, p_1, p_2)$ (and similarly replace $H_0^\pm(\lambda|y-y_1|)$) as in Lemma 3.7. This changes the phase with

$$\phi_\pm(\lambda) = \lambda^2 \pm \lambda \frac{p_2+q_2}{t}.$$

We first consider $\phi_-(\lambda)$, which has a stationary point at $\lambda_0 = \frac{p_2+q_2}{2t} > 0$. We apply Lemma 3.6 with

$$a(\lambda) = \lambda \chi(\lambda) h^\pm(\lambda) \widetilde{G}(\lambda, p_1, p_2) \widetilde{G}(\lambda, q_1, q_2).$$

Using the bounds in Lemma 3.7 (with $\tau = 0+$ for $\widetilde{G}(\lambda, p_1, p_2)$ and $\tau = 0$ for the other), we have

$$(49) \qquad |a(\lambda)| \lesssim \langle x_1 \rangle^{0+} \chi(\lambda) \left(\frac{\widetilde{\chi}(\lambda p_1)}{p_1^{1/2}} + \frac{\widetilde{\chi}(\lambda p_2)}{p_2^{1/2}} \right) \left(\frac{\widetilde{\chi}(\lambda q_1)}{q_1^{1/2}} + \frac{\widetilde{\chi}(\lambda q_2)}{q_2^{1/2}} \right), \quad \text{and}$$

$$(50) \qquad |a'(\lambda)| \lesssim \langle x_1 \rangle \langle y_1 \rangle \frac{\chi(\lambda)}{\lambda} \left(\frac{\widetilde{\chi}(\lambda p_1)}{p_1^{1/2}} + \frac{\widetilde{\chi}(\lambda p_2)}{p_2^{1/2}} \right) \left(\frac{\widetilde{\chi}(\lambda q_1)}{q_1^{1/2}} + \frac{\widetilde{\chi}(\lambda q_2)}{q_2^{1/2}} \right).$$

Let $\tau = 2\max(\lambda_0, t^{-1/2})$. Since $\tilde{\chi}$ is a nondecreasing function supported on $[1, \infty)$ and $\lambda \leq \tau$ on the support of the integral, we have the bound

$$(51) \qquad \int_{|\lambda-\lambda_0|<t^{-1/2}} |a(\lambda)| \, d\lambda \lesssim \langle x_1 \rangle^{0+} t^{-1/2} \left(\frac{\widetilde{\chi}(\tau p_1)}{p_1^{1/2}} + \frac{\widetilde{\chi}(\tau p_2)}{p_2^{1/2}} \right) \left(\frac{\widetilde{\chi}(\tau q_1)}{q_1^{1/2}} + \frac{\widetilde{\chi}(\tau q_2)}{q_2^{1/2}} \right)$$
$$\lesssim \langle x_1 \rangle^{0+} t^{-1/2} \tau \lesssim \langle x_1 \rangle^{0+} t^{-1},$$

if $\tau = 2t^{-1/2}$. On the other hand if $\tau = 2\lambda_0$, we consider the contributions of the products of $\tilde{\chi}$'s more carefully. Consider the contribution of

$$\frac{\widetilde{\chi}(\tau p_i) \widetilde{\chi}(\tau q_j)}{p_i^{1/2} q_j^{1/2}}$$

to (51). This term is zero unless $p_i \gtrsim 1/\lambda_0$ and $q_j \gtrsim 1/\lambda_0$. Therefore, using $p_1 \geq p_2$ and $q_1 \geq q_2$, we have

$$(52) \qquad \frac{p_2+q_2}{p_i q_j} \leq \frac{p_i+q_j}{p_i q_j} = \frac{1}{p_i} + \frac{1}{q_j} \lesssim \lambda_0 = \frac{p_2+q_2}{2t}.$$

Therefore, $p_i q_j \gtrsim t$, and we can estimate the contribution of each product to (51) by $\langle x_1 \rangle^{0+} t^{-1}$.

For the portion of $a(\lambda)$ supported on $|\lambda - \lambda_0| > t^{-\frac{1}{2}}$, we note that if $\lambda_0 \ll t^{-\frac{1}{2}}$, then $|\lambda - \lambda_0| \approx \lambda$ so that

$$\int_{|\lambda-\lambda_0|>t^{-\frac{1}{2}}} \frac{|a(\lambda)|}{|\lambda-\lambda_0|^2} \, d\lambda \lesssim \langle x_1 \rangle^{0+} \sum_{i,j=1}^{2} \int_{\mathbb{R}} \frac{\widetilde{\chi}(\lambda p_i)}{p_i^{1/2}} \frac{\widetilde{\chi}(\lambda q_j)}{q_j^{1/2}} \frac{d\lambda}{\lambda^2}$$
$$\lesssim \langle x_1 \rangle^{0+} \sum_{i,j=1}^{2} \left(\int_{\mathbb{R}} \frac{\widetilde{\chi}(\lambda p_i)^2}{p_i} \frac{d\lambda}{\lambda^2} \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{\widetilde{\chi}(\lambda q_j)^2}{q_j} \frac{d\lambda}{\lambda^2} \right)^{\frac{1}{2}} \lesssim \langle x_1 \rangle^{0+}.$$

On the other hand if $\lambda_0 \gtrsim t^{-\frac{1}{2}}$, we have

$$\int_{|\lambda-\lambda_0|>t^{-\frac{1}{2}}} \frac{|a(\lambda)|}{|\lambda-\lambda_0|^2}\,d\lambda \lesssim \langle x_1\rangle^{0+} \sum_{i,j=1}^2 (p_iq_j)^{-\frac{1}{2}} \int_{|\lambda-\lambda_0|>t^{-\frac{1}{2}}} \frac{\widetilde{\chi}(\lambda p_i)\widetilde{\chi}(\lambda q_j)}{|\lambda-\lambda_0|^2}\,d\lambda.$$

Fix i,j and let $m = \min(p_i,q_j)$. We have two cases: $\lambda_0 \ll 1/m$ and $\lambda_0 \gtrsim 1/m$. In the former case, we note that $|\lambda-\lambda_0| \gtrsim 1/m$ on the support of the cutoffs. Therefore,

$$(p_iq_j)^{-\frac{1}{2}}\int_{|\lambda-\lambda_0|>t^{-\frac{1}{2}}} \frac{\widetilde{\chi}(\lambda p_i)\widetilde{\chi}(\lambda q_j)}{|\lambda-\lambda_0|^2}\,d\lambda \lesssim (p_iq_j)^{-\frac{1}{2}}\int_{|\lambda-\lambda_0|\gtrsim 1/m} \frac{d\lambda}{|\lambda-\lambda_0|^2} \lesssim (p_iq_j)^{-\frac{1}{2}}m \leq 1.$$

In the latter case, using (52), we conclude that $p_iq_j \gtrsim t$. This implies the desired bound by ignoring the cutoffs in the integral.

We now turn to the term in Lemma 3.6 that involves $a'(\lambda)$. Using (50), we have

$$\int_{|\lambda-\lambda_0|>t^{-1/2}} \frac{|a'(\lambda)|}{|\lambda-\lambda_0|}\,d\lambda \lesssim \langle x_1\rangle\langle y_1\rangle \sum_{i,j=1}^2 (p_iq_j)^{-\frac{1}{2}} \int_{|\lambda-\lambda_0|>t^{-1/2}} \frac{\widetilde{\chi}(\lambda p_i)\widetilde{\chi}(\lambda q_j)}{\lambda|\lambda-\lambda_0|}\,d\lambda$$

$$\lesssim \langle x_1\rangle\langle y_1\rangle \sum_{i,j=1}^2 (p_iq_j)^{-\frac{1}{2}} \Big[\int_{|\lambda-\lambda_0|>t^{-1/2}} \frac{\widetilde{\chi}(\lambda p_i)\widetilde{\chi}(\lambda q_j)}{\lambda^2}\,d\lambda + \int_{|\lambda-\lambda_0|>t^{-1/2}} \frac{\widetilde{\chi}(\lambda p_i)\widetilde{\chi}(\lambda q_j)}{|\lambda-\lambda_0|^2}\,d\lambda\Big].$$

The required bounds for each of these terms appeared above in the bound for $a(\lambda)/|\lambda-\lambda_0|^2$ integral. This establishes the desired bound for the phase ϕ_-. For the case of ϕ_+, integration by parts and the bounds on $a(\lambda)$ and $a'(\lambda)$ suffice, we leave the details to the reader. $\square$

With these estimates established, we are ready to prove Proposition 3.2.

Proof of Proposition 3.2. Lemmas 3.4, 3.8, 3.10 and Corollaries 3.5, 3.9 bound each term of (28) as desired. $\square$

We now turn to the terms involving $SS_1D_1S_1$ and $S_1D_1S_1S$ in Corollary 2.7.

Proposition 3.11. *The contribution of the terms $SS_1D_1S_1$, $S_1D_1S_1S$ and QD_0Q in Corollary 2.7 in* (1) *satisfies* (27). *More explicitly, we have the bound*

$$\left|\int_{\mathbb{R}^8}\int_0^\infty e^{it\lambda^2}\lambda\chi(\lambda)R_0^\pm(\lambda^2)(x,x_1)v(x_1)SS_1D_1S_1(x_1,y_1)v(y_1)\right.$$

$$\left. R_0^\pm(\lambda^2)(y,y_1)\,d\lambda f(x)g(y)\,dx_1\,dy_1\,dx\,dy\right| \lesssim |t|^{-1}.$$

The same bound holds when $SS_1D_1S_1$ is replaced by $S_1D_1S_1S$ or by QD_0Q.

Proof. The QD_0Q term can be handled as in Proposition 3.2, it is in fact easier since there is no $\log(\lambda)$ term.

The other terms are somehow different since they have a projection orthogonal to v only on one side. Therefore, one can use (32) only on one side. However, since there is no $\log(\lambda)$ term, the bounds established in in Lemmas 3.4, 3.8, and 3.10 go through. For instance, to establish the bound

$$\left| \int_{\mathbb{R}^8} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \chi(\lambda|x-x_1|) Y_0(\lambda|x-x_1|) v(x_1) SS_1 D_1 S_1(x_1,y_1) v(y_1) \right.$$

$$\left. J_0(\lambda|y-y_1|) \chi(\lambda|y-y_1|) \, d\lambda f(x) g(y) \, dx_1 \, dy_1 \, dx \, dy \right| \lesssim |t|^{-1},$$

we can use $G(\lambda, y, y_1)$ in place of $J_0(\lambda|y-y_1|)$. After an integration by parts the boundary terms vanish since $G(\lambda, y, y_1) \to 0$ as $\lambda \to 0$, and the λ-integral can be bounded by

$$t^{-1} \int_0^\infty \left| \partial_\lambda \left(\chi(\lambda) \chi(\lambda|x-x_1|) Y_0(\lambda|x-x_1|) G(\lambda, y, y_1) \right) \right| d\lambda \lesssim$$

$$t^{-1} \langle y_1 \rangle^\tau (1 + \log^-(|x-x_1|)) \int_0^1 (1 + |\log(\lambda)|) \lambda^{\tau-1} d\lambda \lesssim t^{-1} \langle y_1 \rangle^\tau (1 + \log^-(|x-x_1|)).$$

Here we used the bounds for G from Lemma 3.3, the bounds (6) and (7), and the following estimate:

(53) $$\chi(\lambda) \chi(\lambda|x-x_1|) \log(\lambda|x-x_1|) \lesssim (1 + |\log(\lambda)|)(1 + \log^-(|x-x_1|)).$$

This estimate follows easily by considering the cases $|x-x_1| < 1$ and $|x-x_1| > 1$ separately.

When we have $S_1 D_1 S_1 S$ instead, we must use $F(\lambda, x, x_1)$ instead of $Y_0(\lambda|x-x_1|)$, and the boundary terms are now controlled by $|t|^{-1} k(x, x_1)$ as in Lemma 3.3. The other cases when both projections are onto low energies can be handled similarly.

The case when χ is replaced with $\widetilde{\chi}$ on both sides can be handled as in Lemma 15 from [23] since the argument there does not make use of the projections orthogonal to v.

Similarly, in the case when χ is replaced with $\widetilde{\chi}$ on the side which does not have a projection orthogonal to v, the proof of Lemma 14 from [23] applies.

It remains to prove that

(54) $$\left| \int_{\mathbb{R}^8} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \chi(\lambda|x-x_1|) Y_0(\lambda|x-x_1|) v(x_1) SS_1 D_1 S_1(x_1,y_1) v(y_1) \right.$$

$$\left. J_0(\lambda|y-y_1|) \widetilde{\chi}(\lambda|y-y_1|) \, d\lambda f(x) g(y) \, dx_1 \, dy_1 \, dx \, dy \right| \lesssim |t|^{-1}.$$

Since S is not orthogonal to v, we can not replace χY_0 with F. However, we can replace $\widetilde{\chi} J_0$ with $\widetilde{G}$ shifting the critical point of the λ-integral as in the proof of Lemma 3.8. The argument in the proof of that lemma relies on the bounds

(55) $$|\log(\lambda) F(\lambda, x, x_1)| \lesssim \log(\lambda) k(x, x_1), \quad \left| \partial_\lambda \left(\log(\lambda) F(\lambda, x, x_1) \right) \right| \lesssim k(x, x_1) \log(\lambda) \lambda^{-1}.$$

Since we don't have an additional $\log(\lambda)$ in (54), it suffices to note that $\chi(\lambda) \chi(\lambda|x-x_1|) Y_0(\lambda|x-x_1|)$ satisfies similar bounds as in (55) (with $1 + |\log^-(|x-x_1|)|$ instead of $k(x, x_1)$, c.f., (53)). $\square$

The terms arising from $h_\pm(\lambda)^{-1}S$ and $h_\pm(\lambda)^{-1}SS_1D_1S_1S$ are handled in Lemma 17 in [23], which we restate below for completeness.

Proposition 3.12. [23] *The contribution of the terms $h_\pm(\lambda)^{-1}S$ and $h_\pm(\lambda)^{-1}SS_1D_1S_1S$ in Corollary 2.7 in* (1) *satisfies* (27)*. More explicitly, we have the bound*

$$\left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \left\langle \left[\frac{R_0^+(\lambda^2)vSvR_0^+(\lambda^2)}{h_+(\lambda)} - \frac{R_0^-(\lambda^2)vSvR_0^-(\lambda^2)}{h_-(\lambda)} \right] f, g \right\rangle d\lambda \right| \lesssim |t|^{-1}.$$

A similar bound holds if we replace S with $SS_1D_1S_1S$.

Finally the following proposition (Lemma 18 from [23]) takes care of the contribution of the error term in Corollary 2.7 to (1).

Proposition 3.13. [23] *Assume that $\Phi(\lambda)$ is an absolutely bounded operator on $L^2(\mathbb{R}^2)$ that satisfies $\Phi(\lambda) = \mathcal{O}_1(\lambda^{\frac{1}{2}})$. We have the bound*

$$\left| \int_{\mathbb{R}^8} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) R_0^\pm(\lambda^2)(x, x_1) v(x_1) \Phi(\lambda)(x_1, y_1) v(y_1) R_0^\pm(\lambda^2)(y, y_1) \, d\lambda \right.$$

$$\left. f(x) g(y) \, dx_1 \, dy_1 \, dx \, dy \right| \lesssim |t|^{-1}.$$

4. RESONANCES OF THE SECOND AND THIRD KIND

We now consider the evolution in the case of a p-wave resonance and/or an eigenvalue at zero. Recall that this case is characterized by the non-invertibility of $T_1 = S_1TPTS_1$. To obtain resolvent expansions around zero, we need to invert the operator $B_\pm$, (19). The expansions in this section are considerably more complicated than those in the case of a resonance of the first kind given in Proposition 2.6.

Recall the operators $S_2, S_3, T_2,$ and T_3 from Definition 2.3. With a slight abuse of the notation, we define $D_1 := (T_1 + S_2)^{-1} = (S_1TPTS_1 + S_2)^{-1}$ as an operator on $S_1L^2(\mathbb{R}^2)$. We define $D_2 := (T_2 + S_3)^{-1} = (S_2vG_1vS_2 + S_3)^{-1}$ on $S_2L^2(\mathbb{R}^2)$, we will also use D_2 for T_2^{-1} when T_2 is invertible, i.e. when $S_2 = 0$. We also define $D_3 := T_3^{-1} = (S_3vG_2vS_3)^{-1}$ on $S_3L^2(\mathbb{R}^2)$.

Proposition 4.1. *Assume that $v(x) \lesssim \langle x \rangle^{-3-}$. Then, $B_\pm$ is invertible on $S_1L^2(\mathbb{R}^2)$. In the case of a resonance of the second kind, we have*

$$(56) \qquad\qquad\qquad B_\pm^{-1} = \frac{S_2D_2S_2}{g_1^\pm(\lambda)} + O(\lambda^{-2}|\log\lambda|^{-2}),$$

where $g_1^\pm(\lambda)$ is as in Lemma 2.2.

In the case of a resonance of the third kind, we have

$$(57) \qquad\qquad\qquad B_\pm^{-1} = \frac{S_3D_3S_3}{\lambda^2} + \frac{S_2\mathcal{D}S_2}{g_1^\pm(\lambda)} + O(\lambda^{-2}|\log\lambda|^{-2}),$$

Here $\mathcal{D} = D_2 + S_3D_3S_3vG_2vS_2D_2S_2vG_2vS_3D_3S_3 - S_3D_3S_3vG_2vS_2D_2 - D_2S_2vG_2vS_3D_3S_3.$

Proof. We give the proof for the case of the "+" superscripts and subscripts and omit them from the proof. Recall the definition (19) of B:

$$B = S_1 - S_1(M(\lambda) + S_1)^{-1}S_1.$$

First we repeat the expansion that we obtained in Proposition 2.6 by keeping track of the error term better. Using Lemma 2.5, the identity

$$A^{-1}(\lambda)\big[\mathbb{1} + M_0 A^{-1}(\lambda)\big]^{-1} = A^{-1}(\lambda) - A^{-1}(\lambda)M_0 A^{-1}(\lambda)\big[\mathbb{1} + M_0 A^{-1}(\lambda)\big]^{-1},$$

and the definition (22) of $A^{-1}(\lambda)$, we obtain

$$(58) \qquad B = S_1 - S_1(h(\lambda)^{-1}S + QD_0 Q)S_1 + E(\lambda),$$

where

$$(59) \qquad E(\lambda) = S_1 A^{-1}(\lambda)M_0(\lambda)A^{-1}(\lambda)\big[\mathbb{1} + M_0(\lambda)A^{-1}(\lambda)\big]^{-1}S_1$$

$$= S_1 A^{-1}(\lambda)M_0(\lambda)A^{-1}(\lambda)S_1 - S_1 A^{-1}(\lambda)[M_0(\lambda)A^{-1}(\lambda)]^2\big[\mathbb{1} + M_0(\lambda)A^{-1}(\lambda)\big]^{-1}S_1.$$

Since $v(x) \lesssim \langle x \rangle^{-3-}$, by Lemma 2.2, we have $M_0 = \mathcal{O}_1(\lambda^{2-})$. Also using $A^{-1}(\lambda) = \mathcal{O}_1(1)$ (from (22)), we conclude that

$$(60) \qquad E(\lambda) = \mathcal{O}_1(\lambda^{2-}).$$

Recall that $S_1 D_0 = D_0 S_1 = S_1$. Further, from the definition (21) of S, and the fact that $S_1 P = PS_1 = 0$, we obtain $S_1 S S_1 = S_1 TPTS_1$. Therefore

$$(61) \qquad B = -h(\lambda)^{-1}S_1 SS_1 + E(\lambda) = -h(\lambda)^{-1}S_1 TPTS_1 + E(\lambda).$$

In the case of a resonance of the second kind (unlike the case of a resonance of the first kind), the leading term $T_1 = S_1 TPTS_1$ above is not invertible. We will invert the operator

$$B_1 := -h(\lambda)B = T_1 - h(\lambda)E(\lambda),$$

by using Lemma 2.4. Let S_2 be the Riesz projection onto the kernel of T_1, and let $D_1 := (T_1 + S_2)^{-1}$. We have

$$(B_1 + S_2)^{-1} = [T_1 + S_2 - h(\lambda)E(\lambda)]^{-1} = D_1\big[\mathbb{1} - h(\lambda)E(\lambda)D_1\big]^{-1}$$

$$(62) \qquad = D_1 + D_1 h(\lambda)E(\lambda)D_1 + D_1\big[h(\lambda)E(\lambda)D_1\big]^2\big[\mathbb{1} - h(\lambda)E(\lambda)D_1\big]^{-1} = D_1 + O(\lambda^{2-}).$$

By Lemma 2.4, B_1 is invertible if

$$B_2 := S_2 - S_2(B_1 + S_2)^{-1}S_2$$

is invertible on $S_2 L^2$. Using (62), the identities $S_2 D_1 = D_1 S_2 = S_2$, and the definition (59) of $E(\lambda)$, we have

$$B_2 = -h(\lambda) S_2 E(\lambda) S_2 - S_2 \big[h(\lambda) E(\lambda) D_1 \big]^2 \big[\mathbb{1} - h(\lambda) E(\lambda) D_1 \big]^{-1} S_2$$
$$= -h(\lambda) S_2 A^{-1}(\lambda) M_0(\lambda) A^{-1}(\lambda) S_2 + E_1(\lambda),$$

where

$$(63) \quad E_1(\lambda) = h(\lambda) S_2 A^{-1}(\lambda) [M_0(\lambda) A^{-1}(\lambda)]^2 \big[\mathbb{1} + M_0(\lambda) A^{-1}(\lambda) \big]^{-1} S_2$$
$$- S_2 \big[h(\lambda) E(\lambda) D_1 \big]^2 \big[\mathbb{1} - h(\lambda) E(\lambda) D_1 \big]^{-1} S_2.$$

We now claim that

$$(64) \qquad\qquad\qquad PTS_2 = S_2 TP = 0.$$

To see this, note that since S_2, T and P are self-adjoint, and S_2 is the projection onto the kernel of $S_1 TPTS_1$, we have

$$\langle PTS_2 f, PTS_2 f \rangle = \langle S_2 TPTS_2 f, f \rangle = \langle S_2 S_1 TPTS_1 S_2 f, f \rangle = 0.$$

Therefore,

$$(65) \qquad\qquad\qquad A^{-1}(\lambda) S_2 = S_2 A^{-1}(\lambda) = S_2.$$

Using this and the expansion (14) of M_0, we rewrite B_2 as

$$B_2 = -h(\lambda) g_1(\lambda) S_2 v G_1 v S_2 - h(\lambda) \lambda^2 S_2 v G_2 v S_2 - h(\lambda) S_2 M_1(\lambda) S_2 + E_1(\lambda)$$
$$=: -h(\lambda) g_1(\lambda) \big[T_2 + \lambda^2 g_1^{-1}(\lambda) S_2 v G_2 v S_2 + E_2(\lambda) \big].$$

By Lemma 2.2, we have $M_0 = \mathcal{O}_1(\lambda^{2-})$ and $h(\lambda) S_2 M_1(\lambda) S_2 = \mathcal{O}_1(\lambda^{2-})$. Also using $A^{-1}(\lambda) = \mathcal{O}_1(1)$ and (60), we conclude that $E_1(\lambda) = \mathcal{O}_1(\lambda^{4-})$. This yields that $E_2(\lambda) = \mathcal{O}_1(\lambda^{2-})$. In the case of a resonance of the second kind the leading term is invertible. Therefore, for small λ,

$$(66) \qquad B_2^{-1} = -\frac{\big[T_2 + \lambda^2 g_1^{-1}(\lambda) S_2 v G_2 v S_2 + E_2(\lambda) \big]^{-1}}{h(\lambda) g_1(\lambda)} = -\frac{D_2}{h(\lambda) g_1(\lambda)} + O(\lambda^{-2} |\log \lambda|^{-3}).$$

Using Lemma 2.4, (62), (66), and the identities $S_2 D_1 = D_1 S_2 = S_2$, we have

$$(67) \qquad B^{-1} = -h(\lambda) B_1^{-1} = -h(\lambda) \big[(B_1 + S_2)^{-1} + (B_1 + S_2)^{-1} S_2 B_2^{-1} S_2 (B_1 + S_2)^{-1} \big]$$
$$= \frac{S_2 D_2 S_2}{g_1(\lambda)} + O((\lambda \log \lambda)^{-2}).$$

In the case of a resonance of the third kind, the leading term in B_2 is not invertible. Analogously, we will invert the operator

$$B_3 = -h^{-1}(\lambda)g_1^{-1}(\lambda)B_2 = T_2 + \lambda^2 g_1^{-1}(\lambda)S_2 vG_2 vS_2 + E_2(\lambda)$$

by using Lemma 2.4. Let S_3 be the Riesz projection onto the kernel of T_2, and let $D_2 := (T_2 + S_3)^{-1}$. We have

(68)
$$\begin{aligned}
(B_3+S_3)^{-1} &= \left[T_2 + S_3 + \lambda^2 g_1^{-1}(\lambda)S_2 vG_2 vS_2 + E_2(\lambda)\right]^{-1} \\
&= D_2 - \lambda^2 g_1^{-1}(\lambda)D_2 S_2 vG_2 vS_2 D_2 + \lambda^4 g_1^{-2}(\lambda)D_2[S_2 vG_2 vS_2 D_2]^2 + O(|\log \lambda|^{-3}).
\end{aligned}$$

In the second line we used the definition of $g_1(\lambda)$ in Lemma 2.2 and the estimate on $E_2(\lambda)$.

By Lemma 2.4, B_3 is invertible if

$$B_4 := S_3 - S_3(B_3 + S_3)^{-1}S_3$$

is invertible on $S_3 L^2$. Using (68), the identities $S_3 D_2 = D_2 S_3 = S_3$, and $T_3 = S_3 vG_2 vS_3$, we have

$$B_4 = \lambda^2 g_1^{-1}(\lambda)T_3 - \lambda^4 g_1^{-2}(\lambda)S_3[S_2 vG_2 vS_2 D_2]^2 S_3 + O(|\log \lambda|^{-3}).$$

Since T_3 is always invertible (see Section 4 of [14]), B_4 is invertible for small λ, and we have

$$B_4^{-1} = \lambda^{-2}g_1(\lambda)D_3 + \widetilde{D}_2 + O(|\log \lambda|^{-1}).$$

where $\widetilde{D}_2 = D_3 S_3 vG_2 vS_2 D_2 S_2 vG_2 vS_3 D_3$.

Using this, Lemma 2.4, and (68), we have

$$\begin{aligned}
B_2^{-1} &= -\frac{1}{h(\lambda)g_1(\lambda)}B_3^{-1} = -\frac{1}{h(\lambda)g_1(\lambda)}\left[(B_3 + S_3)^{-1} + (B_3 + S_3)^{-1}S_3 B_4^{-1}S_3(B_3 + S_3)^{-1}\right] \\
&= -\frac{S_3 D_3 S_3}{\lambda^2 h(\lambda)} - \frac{D_2 + S_3\widetilde{D}_2 S_3 - S_3 D_3 S_3 vG_2 vS_2 D_2 - D_2 S_2 vG_2 vS_3 D_3 S_3}{h(\lambda)g_1(\lambda)} + O(\lambda^{-2}|\log \lambda|^{-3}).
\end{aligned}$$

Using this (instead of (66)) for B^{-1} in (67) yields the assertion of the proposition.   □

Corollary 4.2. *Assume that $v(x) \lesssim \langle x \rangle^{-3-}$. Then, in the case of a resonance of the second kind, we have*

(69) $$M^{\pm}(\lambda)^{-1} = \frac{S_2 D_2 S_2}{g_1^{\pm}(\lambda)} + Q\Gamma_1^{\pm}(\lambda)Q + Q\Gamma_2^{\pm}(\lambda) + \Gamma_3^{\pm}(\lambda)Q + \Gamma_4^{\pm}(\lambda)$$

$$+ (M^{\pm}(\lambda) + S_1)^{-1} + \mathcal{O}_1(\lambda^{2-}),$$

where $\Gamma_i^{\pm}$, $i = 1, 2, 3, 4$ are absolutely bounded operators on $L^2(\mathbb{R}^2)$ with $\Gamma_1^{\pm}(\lambda) = O(\lambda^{-2}(\log \lambda)^{-2})$, $\Gamma_2^{\pm}(\lambda), \Gamma_3^{\pm}(\lambda) = O(\lambda^{-2}(\log \lambda)^{-3})$, and $\Gamma_4^{\pm}(\lambda) = O(\lambda^{-2}(\log \lambda)^{-4})$.

In the case of a resonance of the third kind, we have

$$(70) \quad M^\pm(\lambda)^{-1} = \frac{S_3 D_3 S_3}{\lambda^2} + \frac{S_2 \mathcal{D} S_2}{g_1^\pm(\lambda)} + Q\Gamma_1^\pm(\lambda)Q + Q\Gamma_2^\pm(\lambda) + \Gamma_3^\pm(\lambda)Q + \Gamma_4^\pm(\lambda)$$

$$+ (M^\pm(\lambda) + S_1)^{-1} + \mathcal{O}_1(\lambda^{2-}),$$

where $\mathcal{D}$ is as in Proposition 4.1, and Γ_i are absolutely bounded operators on $L^2(\mathbb{R}^2)$. These operators are distinct from the Γ_i in the case of a resonance of the second kind, but satisfy the same size estimates.

Proof. For a resonance of the second kind, combining Proposition 4.1 with Lemma 2.4 and Lemma 2.5 (taking the decay condition on v into account), we have

$$M(\lambda)^{-1} = (M(\lambda) + S_1)^{-1} + (M(\lambda) + S_1)^{-1} S_1 B^{-1} S_1 (M(\lambda) + S_1)^{-1}$$

$$= (M(\lambda) + S_1)^{-1} + \frac{1}{g_1(\lambda)} \big(A(\lambda)^{-1} + \mathcal{O}_1(\lambda^{2-})\big) S_2 D_2 S_2 \big(A(\lambda)^{-1} + \mathcal{O}_1(\lambda^{2-})\big)$$

$$+ \big(A(\lambda)^{-1} + \mathcal{O}_1(\lambda^{2-})\big) O((\lambda \log \lambda)^{-2}) \big(A(\lambda)^{-1} + \mathcal{O}_1(\lambda^{2-}))\big).$$

Using (65) and the definition (22) of A^{-1}, we obtain

$$M(\lambda)^{-1} = \frac{S_2 D_2 S_2}{g_1(\lambda)} + (M(\lambda) + S_1)^{-1} + \mathcal{O}_1(\lambda^{2-})$$

$$+ \big(Q D_0 Q + O(|\log \lambda|^{-1})\big) O((\lambda \log \lambda)^{-2}) \big(Q D_0 Q + O(|\log \lambda|^{-1})\big).$$

The second line leads to four different terms yielding (69).

For the case of a resonance of the third kind, the statement follows similarly using the formula (57) for B^{-1}. $\qquad\qquad\square$

We now consider the dispersive estimates in the case when H has a p-wave resonance at zero energy. Comparing (69) to the expansion in Corollary 2.7, we note that the many of the terms in the expansion for resonances of the second kind are in the expansion for resonances of the first kind. Accordingly, it suffices to establish the estimates for the contributions of the terms:

$$\frac{S_2 D_2 S_2}{g_1^\pm(\lambda)} + Q\Gamma_1(\lambda)Q + Q\Gamma_2(\lambda) + \Gamma_3(\lambda)Q + \Gamma_4(\lambda).$$

We start with the following.

Lemma 4.3. *We have the bound*

$$(71) \quad \left| \int_0^\infty e^{it\lambda^2} \lambda\chi(\lambda) \Big\langle \Big[\frac{R_0^+(\lambda^2)vS_2 D_2 S_2 vR_0^+(\lambda^2)}{g_1^+(\lambda)} - \frac{R_0^-(\lambda^2)vD_2vR_0^-(\lambda^2)}{g_1^-(\lambda)} \Big] f, g \Big\rangle d\lambda \right| \lesssim 1.$$

Proof. We note that we must exploit some cancellation between the '+' and '−' terms. Recall that $H_0^\pm(y) = J_0(y) \pm iY_0(y)$ and the definition of $g_1^\pm(\lambda)$ in Lemma 2.2 give us

$$(72) \quad \frac{R_0^+(\lambda^2)R_0^+(\lambda^2)}{g_1^+(\lambda)} - \frac{R_0^-(\lambda^2)R_0^-(\lambda^2)}{g_1^-(\lambda)} = \frac{J_0(\lambda p)J_0(\lambda q) - Y_0(\lambda p)Y_0(\lambda q)}{\lambda^2[(\log \lambda + c_1)^2 + c_2^2]}$$

$$+ \frac{(J_0(\lambda p)Y_0(\lambda q) + Y_0(\lambda p)J_0(\lambda q))(\log \lambda + c_1)}{\lambda^2[(\log \lambda + c_1)^2 + c_2^2]}$$

We again must use the cut-offs χ and $\widetilde{\chi}$ and consider the different cases depending the supports of the resolvents. Let us first consider the case when both resolvents are supported on low energy. Contribution of the first term in (72) satisfies the required bound since $J_0 = O(1)$, and $\frac{1}{\lambda(\log \lambda)^2}$ is integrable on $[0, \lambda_1]$. Since the other terms have additional powers $\log \lambda$ in the numerator, we need to use (32) (recall that $S_2 \leq Q$).

Consider the contribution of the second term in (72). Using (32), we replace χY_0 with $F(\lambda, \cdot, \cdot)$, and using Lemma 3.3, we obtain the bound:

$$\left| \int_0^\infty \chi(\lambda) \frac{F(\lambda, x, x_1)F(\lambda, y, y_1)}{\lambda[(\log \lambda + c_1)^2 + c_2^2]} \, d\lambda \right| \lesssim k(x, x_1)k(y, y_1) \int_0^1 \frac{1}{\lambda(\log \lambda)^2} \, d\lambda \lesssim k(x, x_1)k(y, y_1).$$

The mixed J_0 and Y_0 terms in the second part of (72) are bounded similarly using $|G(\lambda, x, x_1)| \lesssim \lambda^{0+}\langle x_1 \rangle^{0+}$.

An analysis as in (37) shows that these terms satisfy the desired bound (71).

When one or both of the Bessel functions is supported on high energies, we use the functions $\widetilde{G}(\lambda, p, q)$ from Lemma 3.7. The bound $|\widetilde{G}(\lambda, p, q)| \lesssim \lambda^{0+}|p - q|^{0+}$ suffices for obtaining the required bound. The details are left to the reader. $\qquad \square$

Lemma 4.4. *For $\mathcal{C}_i(z) = J_0(z)$ or $Y_0(z)$ for $i = 1, 2$, we have the bound*

$$\left| \int_{\mathbb{R}^8} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \mathcal{C}_1(\lambda|x - x_1|) v(x_1) Q \Gamma_1(\lambda) Q(x_1, y_1) v(y_1) \mathcal{C}_2(\lambda|y - y_1|) \, d\lambda f(x)g(y) \, dx_1 \, dy_1 \, dx \, dy \right| \lesssim 1.$$

Proof. Unlike in Lemma 4.3 we do not need to use any cancellation between the '+' and '−' terms. We consider the terms that arise when both $\mathcal{C}_1$ and $\mathcal{C}_2$ are supported on small energies. Consider,

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \chi(\lambda p) \mathcal{C}_1(\lambda p) v Q \Gamma_1(\lambda) Q v \chi(\lambda q) \mathcal{C}_2(\lambda q) \, d\lambda,$$

where $p = |x - x_1|$, $q = |y - y_1|$. In the worst case when $\mathcal{C}_1 = \mathcal{C}_2 = Y_0$, using (32), we replace χY_0 with F to obtain

$$(73) \quad \left| \int_0^1 \lambda F(\lambda, x, x_1) \Gamma_1(\lambda) F(\lambda, y, y_1) \, d\lambda \right| \lesssim \left| \int_0^1 \frac{F(\lambda, x, x_1)F(\lambda, y, y_1)}{\lambda(\log \lambda)^2} \, d\lambda \right| \sup_{0 < \lambda < \lambda_1} |\lambda^2(\log \lambda)^2 \Gamma_1(\lambda)|$$

$$\lesssim k(x, x_1)k(y, y_1) \sup_{0 < \lambda < \lambda_1} |\lambda^2(\log \lambda)^2 \Gamma_1(\lambda)|.$$

The last line follows from Lemma 3.3. Since $\sup_{0 < \lambda < \lambda_1} |\lambda^2(\log \lambda)^2 \Gamma_1(\lambda)|$ defines a bounded operator on $L^2(\mathbb{R}^2)$ (by Corollary 4.2), we are done. The other low energy terms are similar using G instead of F from Lemma 3.3.

For the large energies, we note that the argument runs in a similar manner. Using $\widetilde{\chi}(y)(|J_0(y)| + |Y_0(y)|) \lesssim 1$, and an argument as in (73), it easily follows that the integral is bounded as desired. $\qquad \square$

The following modification of Lemma 4.4 is necessary for the other $\Gamma_i(\lambda)$ terms.

Corollary 4.5. *For $\mathcal{C}_i(z) = J_0(z)$ or $Y_0(z)$ for $i = 1, 2$, we have the bound*

$$\left| \int_{\mathbb{R}^8} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \mathcal{C}_1(\lambda|x - x_1|) v(x_1) Q\Gamma_2(\lambda)(x_1, y_1) v(y_1) \mathcal{C}_2(\lambda|y - y_1|) \, d\lambda \, dx_1 \, dy_1 \right| \lesssim 1.$$

The same bounds hold when $Q\Gamma_2(\lambda)$ is replaced by $\Gamma_3(\lambda)Q$ or $\Gamma_4(\lambda)$.

Proof. We repeat the analysis of Lemma 4.4. Consider the case when both $\mathcal{C}_i(\lambda \cdot)$ are supported on low energies and both are Y_0. We note that when $\lambda < 1$, using (53), we have

(74) $$|Y_0(\lambda p)\chi(\lambda p)| \lesssim (1 + |\log \lambda|)(1 + \log^- p).$$

Using this and replacing χY_0 with F on one side, we obtain the bound

$$\int_0^{\lambda_1} \frac{|F(\lambda, x, x_1)|(1 + \log^- q)}{\lambda |\log \lambda|^2} \, d\lambda \sup_{0 < \lambda < \lambda_1} |\lambda^2 (\log \lambda)^3 \Gamma_2(\lambda)| \lesssim k(x, x_1) k(y, y_1) \sup_{0 < \lambda < \lambda_1} |\lambda^2 (\log \lambda)^3 \Gamma_2(\lambda)|.$$

The same bound holds for $\Gamma_3(\lambda)Q$. For the contribution of $\Gamma_4(\lambda)$, we have

$$\left| \int_0^\infty \lambda \chi(\lambda) Y_0(\lambda p) \Gamma_4(\lambda) Y_0(\lambda q) \, d\lambda \right|$$

$$\lesssim \int_0^1 \frac{(1 + |\log \lambda|)(1 + \log^- p)(1 + |\log \lambda|)(1 + \log^- q)}{\lambda |\log \lambda|^4} \, d\lambda \sup_{0 < \lambda < \lambda_1} |\lambda^2 (\log \lambda)^4 \Gamma_4(\lambda)|$$

$$\lesssim k(x, x_1) k(y, y_1) \sup_{0 < \lambda < \lambda_1} |\lambda^2 (\log \lambda)^4 \Gamma_4(\lambda)|.$$

The other cases are similar.

When one of the $\mathcal{C}_i(\lambda \cdot)$ is supported on high energies, the analysis is less delicate. The required bound follows from $\widetilde{\chi}(y)(|J_0(y)| + |Y_0(y)|) \lesssim 1$. □

This completes the proof in the case of a resonance of the second kind.

We note that the above bounds in Lemma 4.4 and Corollary 4.5 also hold for the Γ_i term in (70). Thus for a resonance of the third kind, it suffices to consider the leading λ^{-2} term in (70). Noting (28) and the fact that the kernel of D_3 is real-valued, the following lemma completes the proof. We will prove in the next section that $G_0 v S_3 D_3 S_3 v G_0$ is the projection onto the zero eigenspace whose contribution disappears since we project away from the zero eigenspace. We will ignore this issue in the proof below since the eigenfunctions are bounded functions and hence the projection onto the zero eigenspace satisfies the desired bound, and since removing this operator requires more decay from the potential, see Section 6.

Lemma 4.6. *We have the bound*

(75) $$\left| \int_{\mathbb{R}^8} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) J_0(\lambda|x - x_1|) v(x_1) \frac{S_3 D_3 S_3}{\lambda^2} v(y_1) Y_0(\lambda|y - y_1|) \, d\lambda \, dx_1 \, dy_1 \right| \lesssim 1.$$

Proof. We provide a sketch of the proof. Due to similarities to previous proofs, we leave the details to the reader. We again consider the case when the Bessel functions are supported on low energy first. Accordingly, we wish to control

$$\left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \chi(\lambda p) J_0(\lambda p) v(x_1) \frac{D_3}{\lambda^2} v(y_1) \chi(\lambda q) Y_0(\lambda q) \, d\lambda \right|$$
$$\lesssim \left| \int_0^\infty \chi(\lambda) \frac{G(\lambda, x, x_1) F(\lambda, y, y_1)}{\lambda} \, d\lambda \right| \lesssim \langle x_1 \rangle^\tau k(y, y_1).$$

Where we used (32), Lemma 3.3 with any $\tau > 0$.

For the case when one function is supported on high energy, we have

$$\left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \widetilde{\chi}(\lambda p) J_0(\lambda p) v(x_1) \frac{D_3}{\lambda^2} v(y_1) \chi(\lambda q) Y_0(\lambda q) \, d\lambda \right|$$
$$\lesssim \left| \int_0^\infty \chi(\lambda) \frac{\widetilde{G}(\lambda, x, x_1) F(\lambda, y, y_1)}{\lambda} \, d\lambda \right| \lesssim \langle x_1 \rangle^\tau k(y, y_1).$$

Similarly one uses $\widetilde{G}(\lambda, y, y_1)$ instead of $F(\lambda, y, y_1)$ if we have $\widetilde{\chi}(\lambda q)$.

When both functions are supported on high energy, we have

$$\left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \widetilde{\chi}(\lambda p) J_0(\lambda p) v(x_1) \frac{D_3}{\lambda^2} v(y_1) \widetilde{\chi}(\lambda q) Y_0(\lambda q) \, d\lambda \right|$$
$$\lesssim \left| \int_0^\infty \chi(\lambda) \frac{\widetilde{G}(\lambda, x, x_1) \widetilde{G}(\lambda, y, y_1)}{\lambda} \, d\lambda \right| \lesssim \langle x_1 \rangle^\tau \langle y_1 \rangle^\tau.$$

An analysis as in (37) finishes the proof. □

We are now ready to prove the main theorem of this section.

Theorem 4.7. *Let* $V : \mathbb{R}^2 \to \mathbb{R}$ *be such that* $|V(x)| \lesssim \langle x \rangle^{-\beta}$ *for some* $\beta > 6$. *Further assume that* $H = -\Delta + V$ *has a resonance of the second or third kind at zero energy. Then, there is a time dependent operator* F_t *such that*

$$\sup_t \|F_t\|_{L^1 \to L^\infty} \lesssim 1, \qquad \|K_{\lambda_1} - F_t\|_{L^1 \to L^\infty} \lesssim |t|^{-1}, \quad |t| > 1.$$

Proof. If we denote the terms that arise from the contribution of the terms in the first lines of (69) and (70) as F_t, Lemmas 4.3, 4.4, and 4.6 and Corollary 4.5 show that

$$\sup_t \|F_t\|_{L^1 \to L^\infty} \lesssim 1.$$

As the remaining terms in (69) and (70) are identical in form to those that arise in the proof of Theorem 3.1, we can use the bounds from the previous section to establish the theorem. □

Finally, we note that Theorem 1.1 follows directly from Theorem 1.3, Theorem 3.1, Theorem 4.7, and the first remark following Definition 2.3.

5. Spectral Structure of $-\Delta + V$ at Zero Energy

In this section, we prove some of the claims made in the remark following Definition 2.3. In particular, we show the relationship between the spectral subspaces $S_i L^2(\mathbb{R}^2)$ for $1 = 1, 2, 3$ and distributional solutions to $H\psi = 0$.

Let $w = Uv$. First we characterize $S_1 L^2$.

Lemma 5.1. *If $v \lesssim \langle x \rangle^{-1-}$ and if $\phi \in S_1 L^2$, then $\phi = w\psi$ where $\psi \in L^\infty$, $H\psi = 0$ in the sense of distributions, and*

$$\psi = c_0 - G_0 v\phi, \quad c_0 = \frac{1}{\|V\|_{L^1}} \langle v, T\phi \rangle.$$

Moreover, if $v \lesssim \langle x \rangle^{-2-}$, then $\psi - c_0 = -G_0 v\phi \in L^p$ for any $p \in (2, \infty]$.

Proof. Since $\phi \in S_1 L^2$, we have $Q\phi = \phi$. Using this and $P = I - Q$, we have

$$0 = QTQ\phi = QT\phi = T\phi - PT\phi = U\phi + vG_0 v\phi - PT\phi.$$

Thus,

$$\phi = -wG_0 v\phi + UPT\phi = -wG_0 v\phi + wc_0 = w\psi.$$

Also note that since $v(x) \lesssim \langle x \rangle^{-1-}$ and $\phi \in L^2$, we have $-\Delta G_0(v\phi) = v\phi$. Therefore, we see that $H\psi = 0$ by taking the distributional derivative.

Now we prove that $\psi \in L^\infty$. The boundedness on $B(0, 4)$ is clear. To see that ψ is bounded for $|x| > 4$, use $P\phi = 0$ to obtain

$$G_0 v\phi(x) = -\frac{1}{2\pi} \int [\log(|x - y|) - \log(|x|)] v(y)\phi(y) dy = -\frac{1}{2\pi} \int \log\left(\frac{|x - y|}{|x|}\right) v(y)\phi(y) dy.$$

The bound follows by using the inequality (for $|x| > 4$)

$$(76) \qquad \left| \log\left(\frac{|x - y|}{|x|}\right) \right| \lesssim 1 + \log(\langle y \rangle) + \log^-(|x - y|).$$

Note that this only requires that $v(x) \lesssim \langle x \rangle^{-1-}$.

The final statement follows if we can prove that $G_0 v\phi = O(|x|^{-1})$ for large x. To see this, write

$$G_0 v\phi(x) = -\frac{1}{2\pi} \int \log\left(\frac{|x - y|}{|x|}\right) v(y)\phi(y) dy$$

$$= \int_{|y| > |x|/2} \log\left(\frac{|x - y|}{|x|}\right) v(y)\phi(y) dy + \int_{|y| < |x|/2} \log\left(\frac{|x - y|}{|x|}\right) v(y)\phi(y) dy.$$

The first integral can be estimated by

$$\int_{|y| > |x|/2} [1 + \log(\langle y \rangle) + \log^-(|x - y|)] \frac{|y| |v(y)\phi(y)|}{|x|} dy = O(1/|x|).$$

On the other hand, the bound for the second integral follows from

$$\Big| \log\Big(\frac{|x-y|}{|x|}\Big)\Big| = \Big| \log\Big(1 + \frac{|x-y| - |x|}{|x|}\Big)\Big| = O\Big(\frac{||x-y| - |x||}{|x|}\Big) = O(|y|/|x|).$$

$\square$

Lemma 5.2. *Let $v \lesssim \langle x \rangle^{-2-}$. Assume that the function $\psi = c + \Lambda_1 + \Lambda_2$, with $\Lambda_1 \in L^p$, for some $p \in (2, \infty)$, and $\Lambda_2 \in L^2$, solves $H\psi = 0$ in the sense of distributions. Then $\phi = w\psi \in S_1 L^2$, and we have $\psi = c - G_0 v\phi$, $c = \frac{1}{\|V\|_{L^1}}\langle v, T\phi \rangle$. In particular, by the previous claim, $\psi - c \in L^p$ for any $p \in (2, \infty]$.*

Proof. Since $H\psi = 0$, we have

$$\Delta \psi = V\psi = v\phi.$$

This easily implies that $\int v(y)\phi(y)dy = 0$, see [14, Lemma 6.4]. Thus $\phi \in QL^2$.

Now consider the function $\psi + G_0 v\phi$. By the calculation above, we see that $\Delta(\psi + G_0 v\phi) = 0$. Since $\psi + G_0 v\phi \in L^2 + L^\infty$ (by assumption and the proof of the previous claim), we see that it has to be a constant. Thus

$$\psi = c - G_0 v\phi.$$

Using this, we have

$$TQ\phi = T\phi = U\phi + vG_0 v\phi = U\phi - v\psi + cv = U\phi - U\phi + cv = cv,$$

and hence $QTQ\phi = 0$, and $\phi \in S_1 L^2$. Finally, this implies that $c = \frac{1}{\|V\|_{L^1}}\langle v, T\phi \rangle$. $\square$

Note that Lemma 5.1 and Lemma 5.2 imply that all zero eigenfunctions are bounded. We now characterize $S_2 L^2$.

Lemma 5.3. *Assume that $|v(x)| \lesssim \langle x \rangle^{-3-}$. Then $\phi = w\psi \in S_2 L^2$ if and only if $\psi \in L^p$, for all $p \in (2, \infty]$ (or equivalently $c_0 = 0$).*

Proof. Recall that $S_2 \leq S_1$ is projection onto the kernel of $S_1 TPTS_1$. We have (since $S_1\phi = \phi$)

(77) $$S_1 TPTS_1\phi = 0 \implies \|PT\phi\|^2 = \langle TPT\phi, \phi \rangle = 0,$$

and hence $c_0 = 0$ and $\psi \in L^p$ for $p > 2$.

On the other hand if $\psi \in L^p$ for $p > 2$, we have $c_0 = 0$. This implies that $PT\phi = PTS_1\phi = 0$, and hence $S_1 TPTS_1\phi = 0$. $\square$

Lemma 5.4. *If $v \lesssim \langle x \rangle^{-3-}$ then the kernel of the operator $S_3 vG_2 vS_3$ on $S_3 L^2$ is trivial.*

Proof. Given f in the kernel of $S_3vG_2vS_3$, we have

(78) $$\int_{\mathbb{R}^2} f(y)v(y)\,dy = 0, \text{ and } S_2vG_1vS_2f = 0,$$

since $f \in S_3L^2 \subset QL^2$.

Also note that the expansion we used for $R_0^+(\lambda^2)$ in the proof of Lemma 2.2 gives that

$$R_0^+(\lambda^2)(x,y) = g^+(\lambda) + G_0(x,y) + g_1^+(\lambda)G_1(x,y) + \lambda^2 G_2(x,y) + O(\lambda^{2+}|x-y|^{2+}).$$

This and the assumption $v \lesssim \langle x \rangle^{-3-}$ imply that

$$0 = \langle S_3vG_2vS_3f, f \rangle = \langle vG_2vf, f \rangle = \lim_{\lambda \to 0} \left\langle \frac{R_0^+(\lambda^2) - g^+(\lambda) - G_0 - g_1^+(\lambda)G_1}{\lambda^2}vf, vf \right\rangle.$$

Now, using (78), we have

$$0 = \langle vG_2vf, f \rangle = \lim_{\lambda \to 0} \left\langle \frac{R_0(\lambda^2) - G_0}{\lambda^2}vf, vf \right\rangle = \lim_{\lambda \to 0} \frac{1}{\lambda^2} \int_{\mathbb{R}^2} \left(\frac{1}{4\pi^2\xi^2 + \lambda^2} - \frac{1}{4\pi^2\xi^2} \right) |\widehat{vf}|^2\,d\xi$$

$$= \frac{1}{4\pi^2} \lim_{\lambda \to 0} \int_{\mathbb{R}^2} \frac{1}{\xi^2(4\pi^2\xi^2 + \lambda^2)} |\widehat{vf}|^2\,d\xi = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \frac{|\widehat{vf}|^2}{\xi^4}\,d\xi.$$

Where we used the monotone convergence theorem in the last step. By the assumptions on v and f, $vf \in L^1$, and hence $vf = 0$. We also know that $f \in S_1L^2$ and hence $f = w\psi$, which implies that $f = 0$. This establishes the invertibility of the operator $S_3vG_2vS_3$ on S_3L^2.

Further, we have the identity for any $f \in S_3L^2$,

$$\langle vG_2vf, f \rangle = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \frac{|\widehat{vf}|^2}{\xi^4}\,d\xi = \frac{1}{(2\pi)^4} \left\langle \frac{\widehat{vf}(\xi)}{\xi^2}, \frac{\widehat{vf}(\xi)}{\xi^2} \right\rangle = \langle (-\Delta)^{-1}vf, (-\Delta)^{-1}vf \rangle$$

(79) $$= \langle G_0vf, G_0vf \rangle.$$

$\square$

Lemma 5.5. *Assume that $v(x) \lesssim \langle x \rangle^{-3-}$. Then $\phi = w\psi \in S_3L^2$ if and only if $\psi \in L^2$ with $H\psi = 0$.*

Proof. For $\phi \in S_3L^2 \subset S_1L^2$, we proved above that $\phi = w\psi$, with

$$H\psi = 0, \quad \psi = c_0 - G_0v\phi, \quad c_0 = \frac{1}{\|V\|_{L^1}}\langle v, T\phi \rangle.$$

Also note that $c_0 = 0$ by (77) since $\phi \in S_2L^2$. Therefore, using (79), we have

$$\|\psi\|_2^2 = \langle \psi, \psi \rangle = \langle G_0v\phi, G_0v\phi \rangle = \langle vG_2v\phi, \phi \rangle < \infty$$

by the decay assumption on v.

On the other hand if we assume that $\psi \in L^2$ with $H\psi = 0$, we have that $c_0 = 0$, and hence by Lemma 5.3, we have $\phi = w\psi \in S_2L^2$. We need to prove that $S_2vG_1vS_2\phi = 0$. Note that, as operators on L^2, $S_2vG_1vS_2 = S_2vWvS_2$, where W is the integral operator with kernel $-2x \cdot y$. This is because $G_1(x,y) = |x-y|^2 = |x|^2 - 2x.y + |y|^2$, and the contribution of $|x|^2 + |y|^2$ is zero since $PS_2 = S_2P = 0$.

We claim that if $\psi \in L^2$ with $H\psi = 0$, then

(80)
$$\int_{\mathbb{R}^2} y v(y) \phi(y) \, dy = 0.$$

This implies that

$$S_2 v G_1 v S_2 \phi = S_2 v W v S_2 \phi = -2 S_2 v(x) x \cdot \int_{\mathbb{R}^2} y v(y) \phi(y) \, dy = 0,$$

and hence $\phi \in S_3 L^2$.

It remains to prove the claim above. In what follows below we can assume that $|x| > 4$ since $\psi \in L^\infty$. Define the set $B := \{y \in \mathbb{R}^2 : |y| < |x|/8\}$. Recall that we have $\psi = -G_0 v \phi$, and as $P\phi = 0$ we have

(81)
$$\psi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \ln\left(\frac{|x-y|^2}{|x|^2}\right) v(y)\phi(y) \, dy$$

$$= \frac{1}{4\pi} \int_B \ln\left(\frac{|x-y|^2}{|x|^2}\right) v(y)\phi(y) \, dy + \frac{1}{4\pi} \int_{\mathbb{R}^2 \setminus B} \ln\left(\frac{|x-y|^2}{|x|^2}\right) v(y)\phi(y) \, dy.$$

First we note that the second term is in L^2. Indeed, using (76), and then $1 \lesssim \langle y \rangle / \langle x \rangle$, we see that

$$\left| \int_{\mathbb{R}^2 \setminus B} \ln\left(\frac{|x-y|^2}{|x|^2}\right) v(y)\phi(y) \, dy \right| \lesssim \int_{\mathbb{R}^2 \setminus B} (1 + |y|^{0+} + |x-y|^{0-}) |v(y)\phi(y)| \, dy$$

$$\lesssim \frac{1}{\langle x \rangle^{1+}} \int_{\mathbb{R}^2 \setminus B} \langle y \rangle^{1+} (1 + |y|^{0+} + |x-y|^{0-}) |v(y)\phi(y)| \, dy \lesssim \frac{1}{\langle x \rangle^{1+}} \in L^2(\mathbb{R}^2).$$

We now examine the integral on B. We note that on B, $\left| |y|^2 - 2x \cdot y \right| / |x|^2 < \frac{1}{2}$, and hence

$$\ln\left(\frac{|x-y|^2}{|x|^2}\right) = \ln\left(1 + \frac{|y|^2}{|x|^2} - \frac{2x \cdot y}{|x|^2}\right) = -\frac{2x \cdot y}{|x|^2} + O\left(\frac{\langle y \rangle^{1+}}{\langle x \rangle^{1+}}\right).$$

So that

$$\frac{1}{4\pi} \int_B \ln\left(\frac{|x-y|^2}{|x|^2}\right) v(y)\phi(y) \, dy = -\frac{x}{2\pi |x|^2} \cdot \int_B y v(y)\phi(y) \, dy + O\left(\frac{\int_B \langle y \rangle^{1+} |v(y)\phi(y)| \, dy}{\langle x \rangle^{1+}}\right).$$

The error term is in L^2. We also note that

$$\left| \int_{\mathbb{R}^2 \setminus B} \frac{x \cdot y}{|x|^2} v(y)\phi(y) \, dy \right| \lesssim \int_{\mathbb{R}^2 \setminus B} \frac{\langle y \rangle^{1+}}{\langle x \rangle^{1+}} |v(y)\phi(y)| \, dy \lesssim \langle x \rangle^{-1-} \in L^2(\mathbb{R}^2).$$

Therefore, we can rewrite the main term as

$$-\frac{x}{2\pi |x|^2} \cdot \int_B y v(y)\phi(y) \, dy = -\frac{x}{2\pi |x|^2} \cdot \int_{\mathbb{R}^2} y v(y)\phi(y) \, dy + O_{L^2}(1).$$

Using this in (81), we obtain

$$\psi(x) = \Psi(x) - \frac{x}{2\pi |x|^2} \cdot \int_{\mathbb{R}^2} y v(y)\phi(y) \, dy$$

with $\Psi \in L^2$. As $x/|x|^2$ is not in $L^2(\mathbb{R}^2)$, we must have (80). $\qquad \square$

Lemma 5.6. *The operator $G_0 v S_3 [S_3 v G_2 v S_3]^{-1} S_3 v G_0$ is the orthogonal projection on L^2 onto the zero energy eigenspace of $H = -\Delta + V$.*

Proof. Let $\{\phi_j\}_{j=1}^N$ be an orthonormal basis for the $S_3 L^2$, the range of S_3. Then, we have

(82) $$\phi_j + w G_0 v \phi_j = 0, \qquad 1 \le j \le N.$$

We have $\phi_j = w \psi_j$ for each j with $\psi_j \in L^2$. Since $P S_3 = 0$, we also have

$$\int_{\mathbb{R}^2} V(x) \psi_j(x) \, dx = \int_{\mathbb{R}^2} v(x) \phi_j(x) \, dx = 0.$$

Since $\{\phi_j\}_{j=1}^N$ is linearly independent, we have that $\{\psi_j\}_{j=1}^N$ is linearly independent, and it follows from (82) that

$$\psi_j + G_0 V \psi_j = 0, \quad 1 \le j \le N.$$

Using the orthonormal basis for $S_3 L^2$, we have that for any $f \in L^2(\mathbb{R}^2)$, $S_3 f = \sum_{j=1}^N \langle f, \phi_j \rangle \phi_j$. Therefore, we have

(83) $$S_3 v G_0 f = \sum_{j=1}^N \langle f, G_0 v \phi_j \rangle \phi_j = -\sum_{j=1}^N \langle f, \psi_j \rangle \phi_j.$$

Let $A = \{A_{ij}\}_{i,j=1}^N$ be the matrix representation of $S_3 v G_2 v S_3$ with respect to the orthonormal basis of $S_3 L^2$. Using (79),

$$A_{ij} = \langle \phi_i, S_3 v G_2 v S_3 \phi_j \rangle = \langle G_0 v \phi_i, G_0 v \phi_j \rangle = \langle G_0 V \psi_i, G_0 V \psi_j \rangle = \langle \psi_i, \psi_j \rangle.$$

Let $P_e := G_0 v S_3 [S_3 v G_2 v S_3]^{-1} S_3 v G_0$. Then by (83), for any $f \in L^2(\mathbb{R}^2)$,

$$P_e f = -\sum_{j=1}^N G_0 v S_3 [S_3 v G_2 v S_3]^{-1} \phi_j \langle f, \psi_j \rangle$$

$$= -\sum_{i,j=1}^N G_0 v S_3 (A^{-1})_{ij} \phi_i \langle f, \psi_j \rangle = \sum_{i,j=1}^N \psi_i (A^{-1})_{ij} \langle f, \psi_j \rangle.$$

Note that for $f = \psi_k$, $1 \le k \le N$,

$$P_e \psi_k = \sum_{i,j=1}^N \psi_i (A^{-1})_{ij} \langle \psi_k, \psi_j \rangle = \sum_{i,j=1}^N \psi_i (A^{-1})_{ij} A_{jk} = \psi_k.$$

Thus, we can conclude that the range of P_e is equal to the span of $\{\psi_j\}_{j=1}^N$ and that P_e is the identity on the range of P_e. Since P_e is self-adjoint, the claim is proven. $\qquad \square$

6. A Weighted Estimate

In this section we prove Theorem 1.2. Recall that if zero is an eigenvalue but there are neither s-wave nor p-wave resonances at zero, then $S_1 = S_2 = S_3 \neq 0$. We note that in this case many terms in the expansions of $M^{\pm}(\lambda)^{-1}$ in Corollaries 2.7 and 4.2 disappear. This follows as now

$$(84) \qquad\qquad PS_1 = S_1 P = 0, \quad S_1 TP = PTS_1 = 0, \quad S_1 vG_1 vS_1 = 0.$$

We will also need a finer expansion for $M_0(\lambda)$ then it is given in Lemma 2.2 to prove the theorem. Define $g_2^{\pm}(\lambda) = \lambda^4(a_2 \log \lambda + b_{2,\pm})$ and $g_3(\lambda) = a_3 \lambda^4$ with $a_2, a_3 \in \mathbb{R} \setminus \{0\}$ and $b_{2,-} = \overline{b_{2,+}}$. Also let G_3 be the integral operator with the kernel $|x - y|^4$, and G_4 with the kernel $|x - y|^4 \log |x - y|$. Similar to the expansion given in Lemma 2.2 we obtain

$$(85) \qquad M_0^{\pm}(\lambda) = g_1^{\pm}(\lambda) vG_1 v + \lambda^2 vG_2 v + g_2^{\pm}(\lambda) vG_3 v + g_3(\lambda) vG_4 v + \mathcal{O}_1(\lambda^{9/2}),$$

by expanding the Bessel functions to order $z^6 \log z$ and estimating the error term as in Lemma 2.2. This requires that $|V(x)| \lesssim \langle x \rangle^{-11-}$.

Proposition 6.1. *Assume that $S_1 = S_2 = S_3$, and that $|V(x)| \lesssim \langle x \rangle^{-11-}$. Then, $B_{\pm}$ is invertible on $S_1 L^2(\mathbb{R}^2)$, and we have*

$$(86) \quad B_{\pm}^{-1} = \frac{D_3}{\lambda^2} + \frac{g_2^{\pm}(\lambda)}{\lambda^4} D_3 \Gamma_2 D_3 + D_3 \Gamma_3 D_3 + \frac{g_1^{\pm}(\lambda)^2}{\lambda^4 h^{\pm}(\lambda)} D_3 \Gamma_4 D_3 + \frac{1}{h^{\pm}(\lambda)} D_3 \Gamma_5 D_3$$

$$+ \frac{g_1^{\pm}(\lambda)}{\lambda^2 h^{\pm}(\lambda)} D_3 \Gamma_6 D_3 + \mathcal{O}_1(\lambda^{1/2}),$$

where Γ_i are real-valued, absolutely bounded operators on L^2.

Proof. We will modify the proof of Proposition 4.1. Using (84) in (61) we see that $B = E(\lambda)$ where (from (59))

$$E(\lambda) = S_1 A^{-1}(\lambda) M_0(\lambda) A^{-1}(\lambda) S_1 - S_1 A^{-1}(\lambda)[M_0(\lambda) A^{-1}(\lambda)]^2 \big[\mathbb{1} + M_0(\lambda) A^{-1}(\lambda)\big]^{-1} S_1.$$

Since $S_1 = S_2 = S_3$, using (84) and (22) we have

$$(87) \qquad\qquad S_1 A^{-1}(\lambda) = A^{-1}(\lambda) S_1 = S_3.$$

Using this, Lemma 2.2 and the fact that $A^{-1}(\lambda) = \mathcal{O}_1(1)$, we obtain

$$(88) \qquad B = E(\lambda) = S_3 M_0(\lambda) S_3 - S_3 M_0(\lambda) A^{-1}(\lambda) M_0(\lambda) S_3 + \mathcal{O}_1(\lambda^{9/2}).$$

Using (85) and the fact that $S_3 vG_1 vS_3 = 0$, we get

$$S_3 M_0(\lambda) S_3 = \lambda^2 S_3 vG_2 vS_3 + g_2(\lambda) S_3 vG_3 vS_3 + g_3(\lambda) S_3 vG_4 vS_3 + \mathcal{O}_1(\lambda^{9/2}).$$

We now note that by writing $G_1(x,y) = |x|^2 - 2x \cdot y + |y|^2$, and using (80) and $P \perp Q > S_3$, one obtains

(89) $S_3 v G_1 v Q = Q v G_1 v S_3 = 0.$

Using this and (84) in (22) and (85), we have

$$S_3 M_0(\lambda) A^{-1}(\lambda) M_0(\lambda) S_3 = \frac{g_1(\lambda)^2}{h(\lambda)} S_3 v G_1 v P v G_1 v S_3 + \frac{\lambda^4}{h(\lambda)} S_3 v G_2 v S v G_2 v S_3$$

$$+ \frac{\lambda^2 g_1(\lambda)}{h(\lambda)} [S_3 v G_1 v S v G_2 v S_3 + S_3 v G_2 v S v G_1 v S_3]$$

$$+ \lambda^4 S_3 v G_2 v Q D_0 Q v G_2 v S_3 + \mathcal{O}_1(\lambda^{6-}).$$

Therefore, using these expansions in (88), we have

$$B = \lambda^2 \Gamma_1 + g_2^{\pm}(\lambda) \Gamma_2 + \lambda^4 \Gamma_3 + \frac{g_1^{\pm}(\lambda)^2}{h^{\pm}(\lambda)} \Gamma_4 + \frac{\lambda^4}{h^{\pm}(\lambda)} \Gamma_5 + \frac{\lambda^2 g_1^{\pm}(\lambda)}{h^{\pm}(\lambda)} \Gamma_6 + \mathcal{O}_1(\lambda^{9/2}),$$

where Γ_i are absolutely bounded operators on L^2 with $\Gamma_i = S_3 \Gamma_i S_3$, and $\Gamma_1^{-1} = D_3$. Inverting this via Neumann Series yields the claim of the proposition. $\square$

Corollary 6.2. *Assume that $S_1 = S_2 = S_3$, and that $|V(x)| \lesssim \langle x \rangle^{-11-}$. Then*

(90) $M^{\pm}(\lambda)^{-1} = \dfrac{D_3}{\lambda^2} + (a_1 \log \lambda + b_{1,\pm}) \Xi_1 + \left(1 + \dfrac{b_{3,\pm}}{a_2 \log \lambda + b_{2,\pm}}\right) \Xi_2$

$$+ \frac{1}{h^{\pm}(\lambda)} \Xi_3 + (M^{\pm}(\lambda) + S_1)^{-1} + \mathcal{O}_1(\lambda^{1/2}).$$

Here, Ξ_i are real-valued absolutely bounded operators, Ξ_2 and Ξ_3 have a projection orthogonal to P on at least one side, and Ξ_1 have orthogonal projections on both sides. Further $a_i \in \mathbb{R} \setminus \{0\}$ and $b_{i,+} = \overline{b_{i,-}}$.

We should note that in the statement of the corollary we listed only one term of each form. For example there are several different terms of the form $\frac{b_{3,\pm}}{a_2 \log \lambda + b_{2,\pm}} \Xi_2$ in the expansion.

Proof of Corollary 6.2. Using (23), (22) and (87), and then (84) and (89), we have

$$(M(\lambda) + S_1)^{-1} S_3 = S_3 + A^{-1}(\lambda) M_0(\lambda) S_3 + \mathcal{O}_1(\lambda^{4-})$$

$$= S_3 + \frac{g_1(\lambda)}{h(\lambda)} [P v G_1 v S_3 - Q D_0 Q T P v G_1 v S_3] + \lambda^2 Q D_0 Q v G_2 v S_3 + \frac{\lambda^2}{h(\lambda)} S v G_2 v S_3 + \mathcal{O}_1(\lambda^{4-}),$$

$$S_3 (M(\lambda) + S_1)^{-1} = S_3 + S_3 M_0(\lambda) A^{-1}(\lambda) + \mathcal{O}_1(\lambda^{4-})$$

$$= S_3 + \frac{g_1(\lambda)}{h(\lambda)} [S_3 v G_1 v P - S_3 v G_1 v P T Q D_0 Q] + \lambda^2 S_3 v G_2 v Q D_0 Q + \frac{\lambda^2}{h(\lambda)} S_3 v G_2 v S + \mathcal{O}_1(\lambda^{4-}).$$

Using these and (86) yields that

$$(M(\lambda) + S_1)^{-1} S_3 B^{-1} S_3 (M(\lambda) + S_1)^{-1} = \frac{D_3}{\lambda^2} + (a_1 \log \lambda + b_{1,\pm}) \Xi_1$$

$$+ \left(1 + \frac{b_{3,\pm}}{a_2 \log \lambda + b_{2,\pm}}\right) \Xi_2 + \frac{1}{h^{\pm}(\lambda)} \Xi_3 + \mathcal{O}_1(\lambda^{1/2}).$$

Applying Lemma 2.4 finishes the proof. □

Using Corollary 6.2 and Lemma 5.6 in (26), we see that the contribution of the D_3/λ^2 term can be written as

$$R_0^+(\lambda^2) v \frac{D_3}{\lambda^2} v R_0^+(\lambda^2) = \frac{1}{\lambda^2} \left(R_0^+(\lambda^2) - g^+(\lambda)\right) v D_3 v \left(R_0^+(\lambda^2) - g^+(\lambda)\right)$$

(91)
$$= \frac{1}{\lambda^2} \left(R_0^+(\lambda^2) - g^+(\lambda) - G_0\right) v D_3 v \left(R_0^+(\lambda^2) - g^+(\lambda) - G_0\right)$$

(92)
$$+ \frac{1}{\lambda^2} G_0 v D_3 v \left(R_0^+(\lambda^2) - g^+(\lambda) - G_0\right) + \frac{1}{\lambda^2} \left(R_0^+(\lambda^2) - g^+(\lambda) - G_0\right) v D_3 v G_0$$

(93)
$$+ \frac{1}{\lambda^2} P_e.$$

In the first line above, we used the fact that $PD_3 = D_3 P = 0$ to subtract off $g^+(\lambda)$.

Lemma 6.3. *Under the assumptions of Theorem 1.2, if we project away from the eigenspace of $H = -\Delta + V$ at zero energy, for Schwartz functions f and g the following bound holds.*

$$\left| \int_0^\infty \frac{e^{it\lambda^2} \chi(\lambda)}{\lambda} \langle [R_0^+(\lambda^2) v D_3 v R_0^+(\lambda^2) - R_0^-(\lambda^2) v D_3 v R_0^-(\lambda^2)] f, g \rangle \, d\lambda \right| \lesssim |t|^{-1} \|f\|_{L^{1,1+}} \|g\|_{L^{1,1+}}.$$

Proof. First note that since we project away the zero eigenspace, the contribution of (93) cancels out. To bound the contribution of other terms recall that the expansion for $R_0^\pm(\lambda^2)$ used in Lemma 2.2 gives

$$\left| R_0^+(\lambda^2)(x,y) - g^+(\lambda) - G_0(x,y) \right| \lesssim \lambda^{1+} |x-y|^{1+} \lesssim \lambda^{1+}(\langle x \rangle^{1+} + \langle y \rangle^{1+}),$$

$$\left| \frac{\partial}{\partial \lambda} \left(R_0^+(\lambda^2)(x,y) - g^+(\lambda) - G_0(x,y)\right) \right| \lesssim \lambda^{0+}(\langle x \rangle^{1+} + \langle y \rangle^{1+}).$$

Note that if $|\Phi(\lambda)| + \lambda|\Phi'(\lambda)| \lesssim \lambda^{1+}$ and same for Ψ, then

(94)
$$\left| \int_0^\infty \frac{e^{it\lambda^2} \chi(\lambda)}{\lambda} \Phi(\lambda) \Psi(\lambda) \, d\lambda \right| \lesssim |t|^{-1} \int_0^\infty \left| \frac{d}{d\lambda} \left(\frac{\chi(\lambda) \Phi(\lambda) \Psi(\lambda)}{\lambda^2}\right) \right| d\lambda \lesssim |t|^{-1}.$$

Also using that $|D_3| : L^2 \to L^2$ and $\langle x \rangle^{1+} v(x) \in L^2$, the contribution of (91) satisfies the claim of the lemma.

For the contribution of the terms in (92), we need to use the cancellation between the '+' and '-' terms in Stone's formula.

$$G_0 v D_3 v \left([R_0^+(\lambda^2) - g^+(\lambda)] - [R_0^-(\lambda^2) - g^-(\lambda)]\right)$$

$$= G_0 v D_3 v \left(2i J_0(\lambda | \cdot |) - 2i \Im(z)\right) = 2i G_0 v D_3 v J_0(\lambda | \cdot |).$$

Where we used (32) in the last step. As in the case of an s-wave resonance, we separate into the high and low energies. For the low energy part we use (32) and investigate

$$(95) \qquad \int_0^\infty e^{it\lambda^2} \frac{\chi(\lambda)}{\lambda} \Big(\chi(\lambda|y - y_1|) J_0(\lambda|y - y_1|) - \chi(\lambda(1 + |y_1|)) J_0(\lambda(1 + |y_1|)) \Big) \, d\lambda.$$

After an integration by parts, the result relies on proving the following bound.

$$\int_0^1 \left| \frac{d}{d\lambda} \left(\frac{\chi(\lambda|y - y_1|) J_0(\lambda|y - y_1|) - \chi(\lambda(1 + |y_1|)) J_0(\lambda(1 + |y_1|))}{\lambda^2} \right) \right| d\lambda < C_{y,y_1}.$$

We need not consider when the derivative acts on the cut-off function as this restricts us to an annulus where $\lambda \sim |y - y_1|^{-1}$ or $\lambda \sim (1 + |y_1|)^{-1}$ and we can bound (95) by

$$\int_{\lambda \sim |y-y_1|^{-1}} \frac{1}{\lambda^2} \, d\lambda \lesssim |y - y_1| \lesssim \langle y \rangle \langle y_1 \rangle.$$

The analogous bound holds for $\lambda \sim (1 + |y_1|)^{-1}$. With this in mind, it suffices to prove that

$$\left| \int_0^1 \frac{f(\lambda|y - y_1|) - f(\lambda(1 + |y_1|))}{\lambda^3} \, d\lambda \right| \lesssim \langle y \rangle^{2+} \langle y_1 \rangle^{1+}$$

with $f(z) := \chi(z)[-2J_0(z) + zJ_0'(z)]$. As we are restricted to low energy, we have the expansion for $J_0(z)$ and its derivative in a powers of z from (4). So that

$$f(z) = a_0 + a_1 z + a_2 z^2 + O(z^{2+}).$$

Since $f(0) = -2J_0(0) = -2$ we have that $a_0 = -2$. Further $f'(0) = -J_0'(0) + zJ_0''(z)\big|_{z=0} = 0$, and $f''(z) = zJ_0'''(z)$. Therefore, we have $a_1 = a_2 = 0$, and

$$f(z) = -2 + O(z^{2+}), \qquad f'(z) = O(z^{1+}).$$

Now,

$$\left| f(\lambda|y - y_1|) - f(\lambda(1 + |y_1|)) \right| = \left| \int_{\lambda(1+|y_1|)}^{\lambda|y-y_1|} f'(z) \, dz \right|$$

$$\lesssim \lambda^{2+} \big| |y - y_1| - (1 + |y_1| \big| \big(|y - y_1|^{1+} + \langle y_1 \rangle^{1+} \big) \lesssim \lambda^{2+} \langle y \rangle^{2+} \langle y_1 \rangle^{1+}.$$

Thus, we have

$$(95) \lesssim \left| \int_0^1 \frac{f(\lambda|y - y_1|) - f(\lambda(1 + |y_1|))}{\lambda^3} \, d\lambda \right| \lesssim \langle y \rangle^{2+} \langle y_1 \rangle^{1+} \int_0^1 \lambda^{-1+} \, d\lambda \lesssim \langle y \rangle^{2+} \langle y_1 \rangle^{1+},$$

as desired.

Now, for the high energy we proceed along the lines of the proof of Lemma 3.8. We employ the function $\widetilde{G}^\pm(\lambda, |y - y_1|, 1 + |y_1|)$ of Lemma 3.7. Specifically, we need to bound

$$\int_0^\infty e^{it\lambda^2} \chi(\lambda) G_0 \frac{vD_3 v}{\lambda} \widetilde{G}^\pm(\lambda, p, q) \, d\lambda$$

with $p = \max(|y - y_1|, 1 + |y_1|)$ and $q = \min(|y - y_1|, 1 + |y_1|)$. We will apply Lemma 3.6 to

$$a(\lambda) = \frac{\chi(\lambda)\widetilde{G}^{\pm}(\lambda)}{\lambda}.$$

Using the bounds of Lemma 3.7 with $\tau = 1$, we have

$$|a(\lambda)| \lesssim \langle y \rangle \left(\frac{\widetilde{\chi}(\lambda p)}{|\lambda p|^{1/2}} + \frac{\widetilde{\chi}(\lambda q)}{|\lambda q|^{1/2}} \right) \lesssim \langle y \rangle \left(\widetilde{\chi}(\lambda p) \sqrt{p\lambda} + \widetilde{\chi}(\lambda q) \sqrt{q\lambda} \right),$$

$$|a'(\lambda)| \lesssim \frac{1}{\lambda} \langle y \rangle \left(\frac{\widetilde{\chi}(\lambda p)}{|\lambda p|^{1/2}} + \frac{\widetilde{\chi}(\lambda q)}{|\lambda q|^{1/2}} \right) \lesssim \langle y \rangle \left(\frac{p\widetilde{\chi}(\lambda p)}{|\lambda p|^{1/2}} + \frac{q\widetilde{\chi}(\lambda q)}{|\lambda q|^{1/2}} \right).$$

Here we used that on the support of $\widetilde{\chi}(\lambda p)$, we have $1 \lesssim \lambda p$. At this point, the proof follows exactly along the lines of Lemma 3.8 with the extra weights of $p + q \lesssim \langle y \rangle \langle y_1 \rangle$, which yields the required bound. $\qquad\square$

We are now ready to prove the theorem. We provide a sketch, as there is a significant overlap with the proofs of previous estimates in Section 3.

Proof of Theorem 1.2. We already proved the theorem for the contribution of the D_3/λ^2 term in Corollary 6.2 to (26). The contribution of the Ξ_1 term and the terms in the second line of (90) satisfies the dispersive bound by the results of Section 3. It remains to control the contribution of

$$\left(1 + \frac{b_{3,\pm}}{a_2 \log \lambda + b_{2,\pm}} \right) \Xi_2.$$

We will only provide a brief sketch. Recall that Ξ_2 has projection orthogonal to P only on one side, say on the right. On high energy, we can use $\lambda|x - x_1| \gtrsim 1$ to extract positive powers of λ for the integration at the loss of a weight as in the proof of Lemma 6.3. The polynomial weights arising are either ameliorated by the decay of the potential v or goes into the weight of the weighted dispersive bound. For the low energy part, the worst case is when we have Y_0 on both sides. This arises only with the term containing $\log \lambda$ in the denominator due to the cancellation between the $\pm$ terms. On the right hand side, using (32), we replace χY_0 with F from Lemma 3.3 to reduce to bounding the following integral

$$\left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \chi(\lambda|x - x_1|) Y_0(\lambda|x - x_1|) \frac{b_3}{a_2 \log \lambda + b_2} F(\lambda, y, y_1) \, d\lambda \right|.$$

After an integration by parts it suffices to prove that

$$\int_0^\infty \left| \frac{d}{d\lambda} \left\{ \chi(\lambda) \chi(\lambda|x - x_1|) Y_0(\lambda|x - x_1|) \frac{b_3}{a_2 \log \lambda + b_2} F(\lambda, y, y_1) \right\} \right| d\lambda$$
$$\lesssim k(x, x_1) k(y, y_1) \langle x \rangle^{1+} \langle y \rangle^{1+} \langle x_1 \rangle^{1+} \langle y_1 \rangle^{1+}.$$

We note, from (5), that

$$\chi(\lambda|x - x_1|) Y_0(\lambda|x - x_1|) = \frac{2}{\pi} [\log \lambda + \log |x - x_1| + \gamma] + O(\lambda^{1+}|x - x_1|^{1+}).$$

The first $\log \lambda$ is the most troubling, we note that to control it we use the following facts

$$\left| \log(\lambda) \frac{b_3}{a_2 \log \lambda + b_2} \right| \lesssim 1, \qquad \left| \frac{d}{d\lambda} \left(\log(\lambda) \frac{b_3}{a_2 \log \lambda + b_2} \right) \right| \lesssim \frac{1}{\lambda (\log \lambda)^2}.$$

The contribution of the other terms can be bounded by similar arguments. $\qquad \square$

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